MATH 223 - FINAL EXAM
DECEMBER 2013

Name:
Student ID:

Exam rules:

- No calculators, open books or notes are allowed.
- You do not need to prove results that we proved in class or that appeared in the homework.
- There are 10 problems in this exam. Each problem is worth 6 marks.
- All vector spaces are over real numbers. The notation is the usual one:
  - $\mathbb{R}^n$ – the real n-space.
  - $M_{m\times n}$ – the space of $m \times n$ matrices.
  - Sym_n – the space of $n \times n$ symmetric matrices.
  - P_n – the space of polynomials of degree at most n.
  - A^t is the transpose of the matrix A.
  - $N(T)$ and $R(T)$ are the nullspace and the range of T, respectively.

Good luck!

Problem 1. Consider the system of linear equations $A\vec{x} = \vec{b}$, where

$$A = \begin{bmatrix} 1 & 1 & -1 \\ -1 & c & 2 \\ 1 & 2 & c \end{bmatrix}, \qquad \vec{b} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}.$$

Find all values of c such that the system

(1)has no solution;

(2)has a unique solution;

(3)has infinitely many solutions.

In the last case when there are infinitely many solutions, find all these solutions.

PROBLEM 2. In each part below PROVE that W is a subspace of V and find the dimension of W.

(1)Let $V = M_{n\times n}$, $W = \{\text{all } A \text{ in } V, \text{ such that } \vec{e}_1 \text{ is an eigenvector of } A\}$.

(2)Let $V = M_{2\times 2}$, $W = \{\text{all } A \text{ in } V, \text{ such that } AB = BA\}$. Here

$$B = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}.$$

PROBLEM 3. In each part below PROVE that T is a linear transformation. Find the rank and the nullity of T.

(1)Let $T : P_3 \to M_{2\times 2}$,

$$T(p(x)) = \begin{bmatrix} p(1) & p'(1) \\ p'(2) & p(2) \end{bmatrix}.$$

Here $p'(x)$ is the derivative of $p(x)$.

(2)Let $T : Sym_2 \to M_{2\times 2}$,

$$T(A) = B^t AB.$$

Here

$$B = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}.$$

PROBLEM 4. Find $\det(A^{-1}A^{t}BA^{-1})$, where

$$A = \begin{bmatrix} -2 & 1 & 4 & -1 \\ -1 & 1 & 3 & 1 \\ 5 & -1 & 2 & 1 \\ 2 & -1 & -7 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 & 0 & -4 \\ 0 & 0 & 3 & 5 \\ -1 & 0 & 7 & 2 \\ 4 & 2 & -3 & -7 \end{bmatrix}.$$

Problem 5. Let V be a finite dimensional vector space and $T : V \to V$ a diagonalizable linear transformation.

(1)Prove that

$$Rank(T) = Rank(T^2).$$

(Here T^2 is the composition $T \circ T$.)

(2)Let $\{z_1, \ldots, z_n\}$ be a basis for $N(T)$ and $\{w_1, \ldots, w_m\}$ a basis for $R(T)$. Prove that $\{z_1, \ldots, z_n, w_1, \ldots, w_m\}$ is a basis for V.

Problem 6. Let A be a symmetric matrix with characteristic polynomial $f_A(\lambda) = -\lambda(\lambda - 1)^2$. Assume that

$$\vec{v} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

lies in the nullspace of A.

(1)Find an orthonormal eigenbasis for A.

(2)Find A. (You may leave your answer as a product of matrices.)

Problem 7. In a biology experiment rats are placed in three rooms as shown in the picture. The rats move from room to room using each door with equal probability. A rat in room 1 moves to room 2 with probability 1/2 and to room 3 with probability 1/2 (and stays in room 1 with probability 0.) A rat in room 2 moves to room 1 with probability 1/3 and to room 3 with probability 2/3. A rat in room 3 moves to room 1 with probability 1/3 and to room 2 with probability 2/3.

(1)Find the transition matrix in the Markov chain of this problem.

(2)Find the limiting distribution of rats in each room.

Problem 8. Let $T : P_2 \to \mathbb{R}^3$ be the linear transformation

$$T(p(x)) = \begin{bmatrix} p(2) \\ p'(1) \\ p''(0) \end{bmatrix}.$$

Find the inverse of T. Express the final answer in the form

$$T^{-1}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = (\ldots) + (\ldots)x + (\ldots)x^2.$$

Problem 9. Let A be a $m \times n$ matrix and B a $n \times p$ matrix.

(1)If $AB = 0$ (the zero matrix), prove that

$$Rank(A) + Rank(B) \leq n.$$

(2)If AB has rank r, prove that

$$Rank(A) + Rank(B) \leq n + r.$$

Problem 10. Consider the discrete time dynamical system $\vec{x}_{n+1} = A\vec{x}_n$, where

$$A = \begin{bmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix}, \qquad \vec{x}_0 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}.$$

(1) Express $\vec{x}_0$ in terms of eigenvectors of A. (You may assume that $\lambda = 1$ is one eigenvalue.)

(2) Find $\vec{x}_n$ for arbitrary n.

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