Math 223, Fall Term 2012
Final Exam

December 15th, 2012

Student number:
LAST name:
First name:
Signature:

Instructions

- Do not turn this page over. You will have 150 minutes for the exam (between 12:00–14:30)
- You may not use books, notes or electronic devices of any kind.
- Solutions should be written clearly, in complete English sentences. Proofs should be clear and concise.
- If you are using a result from the textbook, the lectures or the problem sets, state it properly.
- All vector spaces are over the field $\mathbb{R}$ of real numbers unless specified otherwise.

1		/45
2		/15
3		/25
4		/15
Total		/100

1 Calculation

1. (15 points) Let A be the matrix

$$A = \begin{pmatrix} 1 & 2 & 1 & 2 & 1 \\ 3 & 5 & 4 & 3 & 2 \\ 1 & 1 & 2 & -1 & 0 \end{pmatrix}$$

a. Find all solutions to the equation $A\underline{x} = \underline{b}$ where $\underline{b} = \begin{pmatrix} 1 \\ 7 \\ 5 \end{pmatrix}$.

b. Find the rank of A, as well as a basis for the column space.

c. Find the dimension of the nullspace (also called kernel) of A, and a basis for this space.

2. (15 points) Find the eigenvalues and an orthonormal basis of eigenvectors for the following matrix:

$$B = \begin{pmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{pmatrix}$$

You may use the fact that 10 is an eigenvalue.

3. (10 points) Let $\underline{v} = (1,1,1) \in \mathbb{R}^3$. Equip $\mathbb{R}^3$ with its standard inner product.

a. Find a basis for the orthogonal complement $\underline{v}^{\perp}$.

b. Find the matrix (*with respect to the standard basis*) of the orthogonal projection onto $\mathrm{Span}(\underline{v})$.

4. (5 points) For which complex numbers z is the following matrix invertible:

$$\begin{pmatrix} 1 & i & 0 \\ 2 & (1+i) & z \\ 2i & z & (1+i) \end{pmatrix}$$

2 Definition

1. (3 points) Let U, V be vector spaces. Define "*T is a linear map* from U to V" (3 points)

2. (3 points) Let W be another vector space and let $T\colon U \to V$, $S\colon V \to W$ be linear maps. Show that their composition $ST\colon U \to W$ is a linear map as well.

c. (9 points) For each function decide (with proof) whether it is a linear map between the given spaces.

$f_1 \colon \mathbb{R} \to \mathbb{R}$ given by $f_1(x) = x + 1$

$f_2 \colon M_n(\mathbb{R}) \to M_n(\mathbb{R})$ given by $f_2(X) = AXB$ for some fixed $A, B \in M_n(\mathbb{R})$.

$f_3 \colon M_2(\mathbb{R}) \to \mathbb{R}$ given by $f_3(X) = \det(A + X) - \det(A - X)$ where $A \in M_2(\mathbb{R})$ is a fixed matrix.

3 Problems

1. (7 points) Let V be a vector space, and let $\underline{v}_1, \underline{v}_2, \underline{v}_3 \in V$ be three linearly independent vectors in V. Show that the vectors $\underline{v}_1 + \underline{v}_2$, $\underline{v}_1 - \underline{v}_2$, $\underline{v}_1 + \underline{v}_2 + \underline{v}_3$ are linearly independent.

2. (9 points) LetV_n be the space of polynomials of degree less than n, and let $T \in \operatorname{End}(V_n)$ be the map $(Tp)(x) = (x+1)p'(x)$. (For example, $(x^2 - 2x + 5) \in V_3$ and $T\left(x^2 - 2x + 5\right) = 2x^2 - 2$).

a. Find the matrix of T in a basis of V_n (specify which basis you are using). You may wish to analyze small values of n first.

b. Find the eigenvalues of T. Is it diagonalizable?

3. (9 points) For each of the following three possibilities either exhibit a square matrix A satisfying the inequality or show that no such matrix exists.

1. $\text{rank}(A^2) > \text{rank}(A)$;
2. $\text{rank}(A^2) = \text{rank}(A)$;
3. $\text{rank}(A^2) < \text{rank}(A)$.

4 Problem (15 points)

Let V be a vector space, and let $\varphi_1, \ldots, \varphi_k$ some k linear functionals on V. We then have a linear map $\Phi\colon V \to \mathbb{R}^k$ given by $\Phi(\underline{v}) \stackrel{\text{def}}{=} (\varphi_1(\underline{v}), \cdots, \varphi_k(\underline{v}))$ (that is, the ith entry of the vector $\Phi(\underline{v})$ is $\varphi_i(\underline{v})$). Show that Φ is surjective if and only if the φ_i are linearly independent.