**Math 223, Section 101**
**Final Exam**
December 20, 2011
Duration: 150 minutes

Name: ______________________ Student Number: ______________________

**Do not open this test until instructed to do so!** This exam should have 20 pages, including this cover sheet. No textbooks, notes, calculators, or other aids are allowed. Turn off any cell phones, pagers, etc. that could make noise during the exam.

**All your solutions must be written clearly and understandably.** Use complete sentences and explain why your mathematical statements are relevant to the problem. If a result you are quoting requires certain hypotheses, don't forget to write that they are satisfied. You should always write enough to demonstrate that you're not just guessing the answer. Use the backs of the pages if necessary. You will find some of the questions easier than others; solve them in whatever order you like. Good luck!

**Read these UBC rules governing examinations:**

1. Each candidate must be prepared to produce, upon request, a UBCcard for identification.
2. Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
3. No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination.
4. Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action:
   - having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners;
   - speaking or communicating with other candidates; and
   - purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received.
5. Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.
6. Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

| **Problem** | **Out of** | **Score** |
|---|---|---|
| 1 | 6 | |
| 2 | 6 | |
| 3 | 6 | |
| 4 | 9 | |
| 5 | 6 | |
| 6 | 8 | |

| **Problem** | **Out of** | **Score** |
|---|---|---|
| 7 | 7 | |
| 8 | 9 | |
| 9 | 8 | |
| 10 | 7 | |
| 11 | 3 | |
| **Total** | 75 | |

1. Label each of the following statements as either true or false. No justification is required; just choose the correct answer.

(a) **[1 pt]** If a homogeneous system of linear equations has more variables than equations, then it has a nontrivial solution.

**True** ☐ **False** ☐

(b) **[1 pt]** If $M$ is a square matrix such that $\det(M) = -3$, then $\det(M^2) = 9$.

**True** ☐ **False** ☐

(c) **[1 pt]** Let $V$ be an inner product space, and let $v, w \in V$. If $\langle v, w \rangle = 0$, then the set $\{v, w\}$ is linearly dependent.

**True** ☐ **False** ☐

(d) **[1 pt]** If a matrix $K$ is transformed into reduced row echelon form using Gaussian elimination, and so is its transpose $K^t$, then the two resulting reduced row echelon form matrices have the same number of pivots.

**True** ☐ **False** ☐

(e) **[1 pt]** Let $v_1, v_2, v_3 \in \mathbb{R}^3$ (with the standard inner product), and let $P \in M_{3\times 3}(\mathbb{R})$ be the matrix whose three columns are $v_1$, $v_2$, and $v_3$. If $P$ satisfies $P^{-1} = P^t$, then $\{v_1, v_2, v_3\}$ is an orthogonal set.

**True** ☐ **False** ☐

(f) **[1 pt]** Every matrix has an eigenvector.

**True** ☐ **False** ☐

2. Let $V$ be a 7-dimensional vector space, and let $S = \{s_1, s_2, \ldots, s_k\}$ be a subset of $V$. In each situation described below, choose **Always** if $S$ is always a basis for $V$ in that situation; choose **Sometimes** if $S$ is sometimes a basis for $V$ in that situation but sometimes not; choose **Never** if $S$ is never a basis for $V$ in that situation. No justification is required; just choose the correct answer.

(a) **[1 pt]** $k = 7$ (in other words, $S$ has seven vectors).

**Always** ☐ **Sometimes** ☐ **Never** ☐

(b) **[1 pt]** $S$ is linearly independent, and $S$ generates $V$.

**Always** ☐ **Sometimes** ☐ **Never** ☐

(c) **[1 pt]** Every $v \in V$ can be written uniquely as a linear combination of $s_1, s_2, \ldots, s_k$.

**Always** ☐ **Sometimes** ☐ **Never** ☐

(d) **[1 pt]** $s_1$ is a multiple of $s_2$.

**Always** ☐ **Sometimes** ☐ **Never** ☐

(e) **[1 pt]** There exists an invertible linear transformation $T\colon \mathbb{R}^k \to V$ such that $T(e_1) = s_1$, $T(e_2) = s_2$, …, and $T(e_k) = s_k$. (Here $\{e_1, \ldots, e_k\}$ is the standard basis for $\mathbb{R}^k$.)

**Always** ☐ **Sometimes** ☐ **Never** ☐

(f) **[1 pt]** $\mathrm{span}(S)$ is a subspace of $V$. (Remember, the question asks whether $S$ is a basis for $V$ in this situation.)

**Always** ☐ **Sometimes** ☐ **Never** ☐

3. Let $A \in M_{5\times 5}(\mathbb{R})$ be a matrix. In each situation described below, choose **Always** if $A$ is always diagonalizable in that situation; choose **Sometimes** if $A$ is sometimes diagonalizable in that situation but sometimes not; choose **Never** if $A$ is never diagonalizable in that situation. No justification is required; just choose the correct answer.

(a) **[1 pt]** $A$ has $5$ distinct eigenvalues.

**Always** ☐ **Sometimes** ☐ **Never** ☐

(b) **[1 pt]** $A$ is symmetric.

**Always** ☐ **Sometimes** ☐ **Never** ☐

(c) **[1 pt]** $\det(A) = 0$.

**Always** ☐ **Sometimes** ☐ **Never** ☐

(d) **[1 pt]** $A$ has exactly one eigenvalue $\lambda$, and the dimension of its eigenspace is $\dim(E_\lambda) = 1$.

**Always** ☐ **Sometimes** ☐ **Never** ☐

(e) **[1 pt]** $A = B^2$, where $B$ is a diagonalizable matrix.

**Always** ☐ **Sometimes** ☐ **Never** ☐

(f) **[1 pt]** Each of $e_1, e_2, e_3, e_4, e_5$ is an eigenvector for $A$. (Here $\{e_1, \ldots, e_5\}$ is the standard basis for $\mathbb{R}^5$.)

**Always** ☐ **Sometimes** ☐ **Never** ☐

4. Let $\beta = \{1, x, x^2, x^3, x^4\}$ be the standard ordered basis for $P_4(\mathbb{R})$, and let $\gamma = \{E^{11}, E^{12}, E^{21}, E^{22}\}$ be the standard ordered basis for $M_{2\times 2}(\mathbb{R})$. Consider the linear transformation $T\colon P_4(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ defined by the formula

$$T(a + bx + cx^2 + dx^3 + ex^4) = \begin{pmatrix} -b + 7d + 3e & a + b + c + 2d \\ 5a + 9b + c - 6d - 8e & a + 3c + 3d + e \end{pmatrix}.$$

(a) **[2 pts]** Write down the matrix $[T]_\beta^\gamma$.

(b) **[3 pts]** Use Gaussian elimination to transform $[T]_\beta^\gamma$ into reduced row echelon form.

(continued on next page)

Let $\beta = \{1, x, x^2, x^3, x^4\}$ be the standard ordered basis for $P_4(\mathbb{R})$, and let $\gamma = \{E^{11}, E^{12}, E^{21}, E^{22}\}$ be the standard ordered basis for $M_{2\times 2}(\mathbb{R})$. Consider the linear transformation $T\colon P_4(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ defined by the formula

$$T(a + bx + cx^2 + dx^3 + ex^4) = \begin{pmatrix} -b + 7d + 3e & a + b + c + 2d \\ 5a + 9b + c - 6d - 8e & a + 3c + 3d + e \end{pmatrix}.$$

(c) **[2 pts]** Find a basis for the null space $N(T)$.

(d) **[2 pts]** Find a basis for the range $R(T)$.

(this page intentionally left blank for scratch work)

5.

(a) **[4 pts]** Let $A \in M_{4\times 4}(\mathbb{C})$ be given by

$$A = \begin{pmatrix} 1 & 1 & i & 0 \\ 2+3i & 3+3i & -4+4i & 0 \\ 0 & 1 & -2+2i & 0 \\ 0 & 0 & 0 & 4-i \end{pmatrix}.$$

Calculate $A^{-1}$ (the same way you would calculate the inverse of a matrix with real entries). Make it clear what your final answer is, as it will be used in part (b).

(continued on next page)

(b) **[2 pts]** Let $V = M_{2\times 2}(\mathbb{C})$ and $W = \mathbb{C}^4$ be vector spaces over the complex numbers. Let $\beta = \{E^{11}, E^{12}, E^{21}, E^{22}\}$ be the standard ordered basis for $V$, and let $\gamma = \{e_1, e_2, e_3, e_4\}$ be the standard ordered basis for $W$. Define a linear transformation $T\colon V \to W$ by the formula

$$T\left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \big(a + b + ic, (2+3i)a + (3+3i)b + (-4+4i)c, b + (-2+2i)c, (4-i)d\big)$$

and note that $[T]_\beta^\gamma$ equals the matrix $A$ from part (a). (You don't have to prove any of these facts.) Using your answer to part (a), write down the inverse of $T$.

$T^{-1}\big((w, x, y, z)\big) =$

6. Let $T\colon V \to V$ be a linear transformation.

(a) **[2 pts]** Let $g(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$ be a polynomial. Write down the definition of the expression $g(T)$.

(b) **[3 pts]** State the Cayley–Hamilton Theorem for $T$. (Remember that $T$ is a linear transformation, not a matrix.)

(c) **[3 pts]** Prove the Cayley–Hamilton Theorem, assuming that $T$ is diagonalizable.

(this page intentionally left blank for scratch work)

7. In $\mathbb{R}^4$ with the standard inner product, consider the three vectors

$$w_1 = (3, 5, -1, 1), \quad w_2 = (1, 7, 2, 0), \quad w_3 = (-1, 15, 2, 2).$$

(a) **[3 pts]** Perform the Gram–Schmidt process on the set $\{w_1, w_2, w_3\}$ to obtain a new set $\{v_1, v_2, v_3\}$.

(continued on next page)

In $\mathbb{R}^4$ with the standard inner product, consider the three vectors

$$w_1 = (3, 5, -1, 1), \quad w_2 = (1, 7, 2, 0), \quad w_3 = (-1, 15, 2, 2).$$

(b) **[2 pts]** Write down an orthonormal basis for $\operatorname{span}(\{w_1, w_2, w_3\})$.

(c) **[2 pts]** Let $W = \operatorname{span}(\{w_1, w_2\})$. Find vectors $x \in W$ and $y \in W^\perp$ such that $w_3 = x + y$; remember to explain why $x \in W$ and $y \in W^\perp$. (Hint: use your work from part (a).)

8. Let $A = \begin{pmatrix} 2 & 0 & 3 \\ 0 & 1 & 0 \\ -4 & 0 & -5 \end{pmatrix}$.

(a) **[3 pts]** Find the eigenvalues of $A$.

(b) **[4 pts]** Find an invertible matrix $Q$ and a diagonal matrix $D$ such that $Q^{-1}AQ = D$.

(continued on next page)

Let $A = \begin{pmatrix} 2 & 0 & 3 \\ 0 & 1 & 0 \\ -4 & 0 & -5 \end{pmatrix}$.

(c) **[2 pts]** Let $v = \begin{pmatrix} 2 \\ 9 \\ -2 \end{pmatrix}$. Calculate $A^{2011}v$ (this is the $2011$th power of $A$ times $v$). (Hint: use your work from part (b). Either you can calculate $Q^{-1}$, or you might be able to compare $v$ to the columns of $Q$.)

(this page intentionally left blank for scratch work)

9. (The parts of this question aren't that closely related to one another.)

(a) **[3 pts]** Calculate $\det \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 0 & 2 & 0 \\ 3 & 4 & 1 & 2 \\ 3 & 0 & 4 & 0 \end{pmatrix}$.

(continued on next page)

(b) **[2 pts]** Let $A = \begin{pmatrix} r & s & t \\ u & v & w \\ x & y & z \end{pmatrix}$ and $B = \begin{pmatrix} r & 7s & t \\ u & 7v & w \\ x & 7y & z \end{pmatrix}$. What is the relationship between $\det(A)$ and $\det(B)$? Prove that your answer is correct (don't just state a fact from class for this question).

(c) **[3 pts]** Let $U\colon \mathcal{F}(\mathbb{R}^4, \mathbb{R}^5) \to P_6(\mathbb{R})$ be a linear transformation. If $U$ is onto, determine $\mathrm{nullity}(U)$. (You may use the fact that $\mathcal{F}(\mathbb{R}^n, \mathbb{R}^m)$ is isomorphic to $M_{m\times n}(\mathbb{R})$ for any positive integers $m$ and $n$.)

10. Let $V$ be an inner product space.

(a) **[2 pts]** Let $v, w \in V$, and suppose that $v$ is orthogonal to $w$. Prove that

$$\|v + w\| = \sqrt{\|v\|^2 + \|w\|^2}.$$

(b) **[3 pts]** Let $S$ be a subset of $V$. Prove that $S^\perp$ is a subspace of $V$.

(c) **[2 pts]** Let $T \colon V \to V$ be a linear operator, and suppose that $\|T(x)\| = \|x\|$ for every $x \in V$. Prove that $T$ is one-to-one. (Hint: null space.)

11. **[3 pts]** Let $A \in M_{4\times 4}(\mathbb{R})$ be a matrix with the following properties:

- $\mathrm{rank}(A) = 2$;
- $\mathrm{rank}(A - 5I_4) = 3$;
- The trace of $A$ is $\mathrm{tr}(A) = 10$.

Prove that $A$ is *not* diagonalizable.