1. (10 points) Let W denote the subspace of $\mathbb{R}^4$ spanned by the set

$$\{(1,1,0,0),(1,0,1,0),(0,1,0,1),(0,0,1,1)\}.$$

Find an orthogonal basis for W.

2. (10 points) Let X be a subspace of $\mathbb{R}^n$. Show that there exists a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ such that the range of T is exactly X.

3. (15 points) Let T be the matrix

$$T = \begin{pmatrix} 2 & 0 & 0 \\ 2 & 6 & 0 \\ 3 & 2 & 1 \end{pmatrix}.$$

(a) Find all real eigenvalues for T.

(b) Say whether T is diagonalizable or not.

(c) If T is diagonalizable, find an invertible matrix P such that $P^{-1}TP$ is diagonal.

4. (10 points) Is the matrix

$$T = \begin{pmatrix} 2 & 0 & 0 \\ 2 & 6 & 0 \\ 3 & 2 & 1 \end{pmatrix}$$

from the previous problem invertible? If so compute its inverse.

5. (10 points) Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation with matrix

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 11 & 14 & 17 & 20 \end{pmatrix}.$$

Find a basis for the null-space of T. Then find a basis for the range of T.

6. (10 points) Determine whether $S = \{(1,1,1,1),(1,2,3,2),(2,5,6,4),(2,6,8,5)\}$ is a linearly independent subset of $\mathbb{R}^4$. If not, write one of the elements of S as a linear combination of the others.

7. (10 points) Suppose

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is a real 2×2 matrix whose trace $a + d$ is 1 and whose determinant is 0. Show that $A^2 = A$.

8. (10 points) Let n be a positive integer. For each integer $i \in [1,n]$, let $p_i : \mathbb{R}^n \to \mathbb{R}^{n-1}$ denote the linear map $p_i(x_1,\dots,x_n) = (x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)$. Suppose $W \subset \mathbb{R}^n$ is a proper subspace. Show that there is an integer i such that the map $\pi_i : W \to \mathbb{R}^{n-1}$ given by $w \mapsto p_i(w)$ is injective. (Recall that a proper subspace of $\mathbb{R}^n$ is a subspace which is not all of $\mathbb{R}^n$.)

9. (15 points) Let V be a vector space over a field F and let k be a positive integer. A *flag* in V is a sequence

$$V_0 \subset V_1 \subset \cdots \subset V_k$$

of subspaces of V such that, for each integer $i \in [0, k-1]$, V_i is a proper subspace of V_{i+1}. The integer k is called the *length* of the flag. For example, if $V = \mathbb{R}^3$, then

$$\{0\} \subset \langle\{(1,0,0),(1,2,0)\}\rangle \subset V$$

is a flag.

(a) Write down a flag of length n in the vector space F^n.

(b) Suppose that $f : V \to W$ is an injective linear map of F vector spaces. And $V_0 \subset V_1 \subset \cdots \subset V_k$ is a flag in V. Show that $f(V_0) \subset f(V_1) \subset \cdots \subset f(V_k)$ is a flag in W.

(c) Show that the the biggest possible length of a flag in F^2 is 2.

Math 223 Final Exam Spring 2009

Patrick Brosnan, Instructor

First Name/Last Name: ______________________

Student ID Number: ______________________

Section/Professor: ______________________

Signature:

By signing here, you confirm you are the person identified above and that all the work herein is solely your own.

Instructions:

(1) No calculators, books, notes, or other aids allowed.
(2) Give your answer in the space provided. If you need extra space, use the back of the page. **PLEASE BOX ALL FINAL ANSWERS!** And **clearly indicate whether you are planning to prove a statement or give a counterexample at the beginning of the problem.**
(3) Show enough of your work to justify your answer. Show ALL steps.

Problem	Points	Score
1	10	
2	10	
3	15	
4	10	
5	10	
6	10	
7	10	
8	10	
9	15	
Total	100	