MATH 223 - FINAL EXAM
DECEMBER 2007

Name:
Student ID:

Exam rules:

- No calculators, open books or notes are allowed.
- You do not need to prove results that we proved in class or that appeared in the homework.
- There are 10 problems in this exam. Each problem is worth 5 marks, except problems 1 and 2 where each part is worth 2 marks.
- All vector spaces are over real numbers. The notation is the usual one:
  - $\mathbb{R}^n$ – the real n-space.
  - $M_{m\times n}$ – the space of $m \times n$ matrices.
  - P_n – the space of polynomials of degree at most n.
  - A^t is the transpose of the matrix A.
  - $N(T)$ and $R(T)$ are the nullspace and the range of T, respectively.

Please draw a box around your final answer to each problem.

Good luck!

Problem 1. In each part below determine if W is a subspace of V, and if it is, find the dimension of W. No proofs are needed here. (But you may want to write down a proof anyway to convince yourself.)

(1)Let $V = Mat_{n\times n}$, $W = \{A \in V | A\vec{e}_1 = \vec{e}_1\}$.

(2)Let $T : \mathbb{R}^m \to \mathbb{R}^n$ be an onto linear transformation and $U \subset \mathbb{R}^n$ a subspace of dimension k. Let

$$V = \mathbb{R}^m, \qquad W = \{\vec{v} \in V | T(\vec{v}) \in U\}.$$

(3)Let $\vec{v}_1, \ldots, \vec{v}_n$ be a set of vectors in $\mathbb{R}^m$ that spans $\mathbb{R}^m$. Then

$$V = \mathbb{R}^n, \qquad W = \{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \in V | a_1\vec{v}_1 + \ldots + a_n\vec{v}_n = \vec{0} \}.$$

(Hint: construct a linear transformation $\mathbb{R}^n \to \mathbb{R}^m$.)

(4)Let V be the set of all sequences $(a_1, a_2, \ldots)$ with addition and scalar multiplication componentwise as in $\mathbb{R}^n$. Let W consist of all sequences satisfying $a_n = a_{n-1} + a_{n-2} + 1$ for all $n \geq 3$.

(5) Assume that $\beta = \{\vec{v}_1, \dots, \vec{v}_n\}$ is a basis of $\mathbb{R}^n$. Let $V = Mat_{n\times n}$ and W the set of matrices that have β as an eigenbasis.

(6) Let $V = Mat_{n\times n}$, and let $A \in V$ be a matrix of rank r. Then

$$W = \{B \in V | AB = 0\}$$

(Hint: Think in terms of linear transformations.)

PROBLEM 2. In each part below determine if T is a linear transformation. If it is linear, find the rank and the nullity of T. No proofs are needed.

(1)Let $T : P_3 \to P_3$, $T(p(x)) = x^3 p(\frac{1}{x})$.

(2)Let $T : \mathbb{R}^n \to \mathbb{R}$, $T(\vec{v}) = |\vec{v}|$, where $|\vec{v}|$ is the length of $\vec{v}$.

(3)Let $T : P_3 \to P_4$,

$$T(p(x)) = \int_0^x p(t)dt.$$

(4)Let $T : Mat_{3\times 3}$, $T(A) = A + 2A^t$.

Problem 3. Find all solutions to the system of linear equations $A\vec{x} = \vec{b}$, where

$$A = \begin{bmatrix} 1 & 1 & 0 & 5 & 0 & -1 \\ 0 & 1 & 1 & 3 & -2 & 0 \\ -1 & 2 & 3 & 4 & 1 & -6 \\ 0 & 4 & 4 & 12 & -1 & -7 \end{bmatrix}, \qquad \vec{b} = \begin{bmatrix} 3 \\ -1 \\ 1 \\ 3 \end{bmatrix}.$$

PROBLEM 4. Consider the sequence of numbers $0, 1, 2, 5, 12, \ldots$, where $a_{n+1} = 2a_n + a_{n-1}$. When n is large, then a_{n+1} is approximately $c \cdot a_n$. Find the constant c.

Problem 5. Let $T : V \to W$ and $S : W \to V$ be linear transformations such that $S \circ T = Id_V$. Let $\vec{v}_1, \ldots, \vec{v}_n$ be a basis of V and $\vec{w}_1, \ldots, \vec{w}_m$ a basis for $N(S)$. Prove that

$$T(\vec{v}_1), \ldots, T(\vec{v}_n), \vec{w}_1, \ldots, \vec{w}_m$$

forms a basis of W.

PROBLEM 6. Find the determinant of the matrix

$$\begin{bmatrix} 1 & -1 & 5 & 5 \\ 3 & 1 & 2 & 4 \\ -1 & -3 & 8 & 0 \\ 1 & 1 & 2 & -1 \end{bmatrix}$$

Problem 7. Let A be a symmetric matrix with eigenvalues 2 and 6. If the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

span E_6, find $A \cdot \vec{e}_1$.
(Hint: Find a third eigenvector and expand $\vec{e}_1$ in the eigenbasis.)

Problem 8. Consider the following population model of counting a certain species of birds. Divide the total population in year k into two groups: j_k is the number of juvenile birds and a_k the number of adult birds. A newly hatched bird remains juvenile for one year and then becomes an adult (in other words, a bird hatched in year k counts as a juvenile in year k, and as an adult in year $k+1$.). The following rules describe how to compute j_{k+1} and a_{k+1}:

- $\frac{1}{2}$ of adults survive to the next year.
- $\frac{1}{4}$ of juveniles survive to the next year to become adults.
- The number of juveniles hatched in year $k+1$ is twice the number of adults in year k.

Given initial populations $j_0 = 3, a_0 = 3$ (in thousands), find the limit of j_k, a_k as k approaches infinity. (Hint: Express the initial population in terms of an eigenbasis.)

Problem 9. Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

(To check your computation, the inverse of a symmetric matrix is also symmetric.)

PROBLEM 10. Find an orthonormal eigenbasis for the matrix

$$\begin{bmatrix} 8 & -2 & 2 \\ -2 & 5 & 4 \\ 2 & 4 & 5 \end{bmatrix}.$$

You may assume that the characteristic polynomial of the matrix is $-\lambda^3+18\lambda^2-81\lambda$.

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