Be sure this exam has 3 pages.

THE UNIVERSITY OF BRITISH COLUMBIA

Sessional Examination - December 2006

MATH 223: Linear Algebra

Instructor: Dr. R. Anstee, section 101

Special Instructions: No Aids. No calculators or cellphones. 3 hours

You must show your work and explain your answers.

1. [16 marks] Consider the matrix equation $A\mathbf{x} = \mathbf{b}$ with

$$A = \begin{bmatrix} 1 & -1 & 0 & 1 & 1 & 0 \\ 2 & 0 & 4 & 4 & 2 & 2 \\ 2 & -1 & 2 & 3 & 2 & 1 \\ 1 & 1 & 4 & 3 & 0 & 3 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 4 \\ 3 \\ 2 \end{bmatrix}$$

There is an invertible matrix B so that

$$BA = \begin{bmatrix} 1 & -1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad B\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

a) [1 mark] What is rank(B)?
b) [2 marks] What is rank(A)?
c) [4 marks] Give the vector parametric form for the set of solutions to $A\mathbf{x} = \mathbf{b}$.
d) [6 marks] Give a basis for the row space of A. Give a basis for the column space of A. Give a basis for the null space of A.
e) [3 marks] How many linearly independent vectors $\mathbf{b}'$ can you find so that the system of equations $A\mathbf{x} = \mathbf{b}'$ is consistent?

2. [15 marks] Let

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 0 \end{bmatrix}$$

Determine an orthonormal basis of eigenvectors and hence an orthogonal matrix Q and a diagonal matrix D so that $A = QDQ^T$. You may find it useful to know that 5 is an eigenvalue of A.

3. [9 marks]
a) [3 marks] What is the distance of the point $(1, 1, 1)^T$ from the plane $x - 2y + 3z = 1$? Note that the plane does not go through the origin.

b) [3 marks] We are given that A is a symmetric invertible matrix. Show that A^{-1} is symmetric.

c) [3 marks] Let P, Q be orthogonal matrices. Show that Q^2P^T is an orthogonal matrix.

4. [15 marks] Let

$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

You may find it useful to note that:

$$\begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 1 & 2 \end{bmatrix}$$

a) [1 marks] Explain why $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ forms a basis for $\mathbf{R}^3$.

b) [5 marks] Consider the linear transformation $T : \mathbf{R}^3 \to \mathbf{R}^3$ satisfying

$$T(\mathbf{u}_1) = \mathbf{u}_2 + \mathbf{u}_3, \quad T(\mathbf{u}_2) = \mathbf{u}_2 + 2\mathbf{u}_3, \quad T(\mathbf{u}_3) = 3\mathbf{u}_2 + \mathbf{u}_3.$$

Give the matrix A representing T with respect to the basis $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ (both input vector and output vector).

c) [5 marks] Give the matrix B representing T where the input vector $\mathbf{x}$ is written with respect to the basis $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ (i.e. written in $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$-coordinates) and the output vector $A\mathbf{x}$ is written with respect to the standard basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ (i.e. written in $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$-coordinates).

d) [4 marks] What is rank(A)? Explain why rank$(A) =$ rank(B), even for a different choice of T?

5. [6 marks] Let A be a 3×3 matrix with eigenvalues $2, 4, 6$. What are the eigenvalues of $A + 2I$?

6. [7 marks] Determine the matrix A corresponding to the orthogonal projection into the plane $x - 2y + 2z = 0$.

7. [10 points] Consider the system of differential equations:

$$\frac{d}{dt}x_1(t) = 2x_1(t) + x_2(t).$$

$$\frac{d}{dt}x_2(t) = -2x_1(t)$$

You will find it useful to be given that

$$\begin{bmatrix} 2 & 1 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1-i & 1+i \end{bmatrix} \begin{bmatrix} 1+i & 0 \\ 0 & 1-i \end{bmatrix} \begin{bmatrix} -\frac{1}{2}+\frac{1}{2}i & \frac{1}{2}i \\ -\frac{1}{2}-\frac{1}{2}i & -\frac{1}{2}i \end{bmatrix},$$

$$\begin{bmatrix} -1 & -1 \\ 1-i & 1+i \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{1}{2}+\frac{1}{2}i & \frac{1}{2}i \\ -\frac{1}{2}-\frac{1}{2}i & -\frac{1}{2}i \end{bmatrix}$$

Find the solution to the system of differential equations that satisfies $x_1(0) = x_2(0) = 1$. For full marks you must simplify your answer so no complex numbers appear.

8. [12 marks]
   a) [4 marks] Given two vectors $\mathbf{u}_1 = (1, 2, 2, 0)^T$ and $\mathbf{u}_2 = (0, 3, 6, 0)^T$, find two vectors $\mathbf{u}_3, \mathbf{u}_4$ so that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ is a basis for $\mathbf{R}^4$.
   b) [4 marks] Given the two vectors $\mathbf{u}_1$, $\mathbf{u}_2$ as in a), apply Gram-Schmidt to obtain an orthonormal basis for the vector space $V = \text{span}\{\mathbf{u}_1, \mathbf{u}_2\}$.
   c) [4 marks] Use the orthonormal basis for $V = \text{span}\{\mathbf{u}_1, \mathbf{u}_2\}$ from b) to express $\mathbf{v} = (1, 1, 1, 1)^T$ as a sum $\mathbf{v} = \mathbf{u} + \mathbf{w}$ where $\mathbf{u} \in V$ and $\mathbf{w} \in V^{\perp}$. You should not need to compute a complete orthonormal basis for $\mathbf{R}^4$.

9. [10 marks] Let A be a symmetric 4×4 matrix with $\det(A - \lambda I) = (\lambda - 2)(\lambda + 1)^3$. Assume that $(1, 1, 1, 1)^T$ is an eigenvector of eigenvalue 2. Show that $(1, -1, 0, 0)^T$ is an eigenvector of eigenvalue -1.

100 Total marks