MATH 223 – FINAL EXAM
APRIL, 2005

Instructions:

(a) There are 10 problems in this exam. Each problem is worth five points, divided equally among parts.

(b) Full credit is given to complete work only. Simply writing down an answer is not enough (unless told otherwise).

(c) No calculators, books or notes are allowed.

(d) All vector spaces are assumed to be real vector spaces. No complex numbers will be needed.

(e) The usual notation is assumed:

- $M_{m\times n}$ is the space of $m \times n$ matrices.
- P_n is the space of polynomials in one variable of degree n or less.
- P is the space of all polynomials in one variable.

(f) Good luck!

Problem 1. Which of the following sets W are subspaces? No proof is necessary, although you may want to prove it to convince yourself.

(a) $W \subset P_4$ the set of palindromic polynomials:

$$W = \{a + bx + cx^2 + bx^3 + ax^4 | a, b, c \in \mathbb{R}\}.$$

(b) $W \subset M_{2\times 2}$ the set of non-invertible matrices:

$$W = \{A \in M_{2\times 2} | \det(A) = 0\}.$$

(c) Given linear transformations $T : U \to V$ and $S : U \to V$, let $W \subset V$ be the set of all vectors $\vec{v} \in V$ that can be expressed as

$$\vec{v} = T(\vec{u}_1) + S(\vec{u}_2)$$

for some vectors $\vec{u}_1, \vec{u}_2 \in U$.

(d) $W \subset \mathbb{R}^3$ is the intersection of two cylinders defined by the equations $(x-1)^2 + y^2 = 1$ and, $(x+1)^2 + y^2 = 1$:

$$W = \{(x, y, z)| \quad (x-1)^2 + y^2 = 1, \quad (x+1)^2 + y^2 = 1\}.$$

(Hint: draw a picture of the cylinders.)

(e) $W \subset M_{n\times n}$ the set of all matrices having the vector

$$\vec{v} = \begin{bmatrix} 1 \\ 2 \\ \vdots \\ n \end{bmatrix}$$

as an eigenvector (with arbitrary eigenvalue).

PROBLEM 2. Which of the following functions T are linear transformations? Again, no proof is necessary.

(a) $T : P_3 \to P_6$ is given by $T(f(x)) = f(x^2)$.

(b) $T : M_{3\times 3} \to M_{3\times 3}$ that adds the first column to the last one:

$$T[\vec{a}_1|\vec{a}_2|\vec{a}_3] = [\vec{a}_1|\vec{a}_2|\vec{a}_1 + \vec{a}_3].$$

(c) $T : P \to \mathbb{R}$ that maps the zero polynomial to zero and a nonzero polynomial to its last nonzero coefficient:

$$T(a_0 + a_1 x + \ldots + a_n x^n) = a_n, \quad \text{where } a_n \neq 0.$$

(d) $T : M_{n\times n} \to P_n$ that maps a matrix A to its characteristic polynomial $f_A(t)$.

(e) $T : \mathbb{R} \to \mathbb{R}$ given by $T(x) = 5x + 3$.

PROBLEM 3. Note that if $\vec{v}, \vec{w} \in \mathbb{R}^n$ are column vectors, then the product of matrices $\vec{v} \cdot \vec{w}^t$ is an $n \times n$ matrix. Let

$$\vec{w} = \begin{bmatrix} 1 \\ 2 \\ \vdots \\ n \end{bmatrix}.$$

If $\vec{v}_1, \ldots, \vec{v}_n$ is a basis of $\mathbb{R}^n$, show that the set of matrices

$$S = \{\vec{v}_1 \cdot \vec{w}^t, \vec{v}_2 \cdot \vec{w}^t, \ldots, \vec{v}_n \cdot \vec{w}^t\}$$

is linearly independent. (Hint: What are the columns of the matrices in S?)

Problem 4. Let $T : \mathbb{R}^5 \to \mathbb{R}^4$ be the linear transformation

$$T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_1 - 2x_2 + 2x_3 + 4x_5 \\ -x_1 + 2x_2 + 2x_3 + x_4 + 7x_5 \\ x_3 + 3x_4 \\ 2x_1 - 4x_2 - x_3 + 3x_4 - 10x_5 \end{bmatrix}.$$

(a) Find a basis for the null-space of T.

(b) Find a basis for the range of T.

Problem 5. Let V be the vector space of all continuous functions $f : \mathbb{R} \to \mathbb{R}$ and let $W \subset V$ be the subspace spanned by

$$e^x, xe^x, x^2e^x.$$

You may assume without proof that these three functions are linearly independent. Let $T : W \to W$ be the derivative d/dx. Find the inverse of T. (Note: you can find the inverse by inverting the matrix of T in some basis, but your final answer should be in the form

$$T^{-1}(ae^x + bxe^x + cx^2e^x) = (a + b + 2c)e^x + \ldots + (\ldots)x^2e^x.$$

Since T is the derivative, its inverse can be found by integration. However, only minimal credit will be given for such a calculus proof.)

Problem 6. Let A be a nonzero 10×10 matrix such that $A^{25} = 0$.

(a) Show that 0 is an eigenvalue of A; in other words, there is a corresponding eigenvector $\vec{v}$.

(b) Show that A has no other eigenvalues.

(c) Show that A is not diagonalizable.

PROBLEM 7. A co-op produces four types of cookies: A, B, C, and D. Ingredients needed to make one box of cookies of each type are given in the table below:

	A	B	C	D
flour (cups)	3	6	3	9
butter (lb.)	1	2	1	3
sugar (cups)	2	1	3	8
eggs	1	4	2	7
chocolate (lb.)	0	3	1	2

During one hour of operation, the following amount of ingredients were used: 33 cups of flour, 11 lb. of butter, 23 cups of sugar, 21 eggs, 7 lb. of chocolate. Find how many boxes of each type were produced.

PROBLEM 8. Let A_n be the $n \times n$ matrix below with non-zero entries on the three diagonals only:

$$A_n = \begin{bmatrix} 6 & 1 & & & & \\ 5 & 6 & 1 & & & \\ & 5 & 6 & 1 & & \\ & & & \dots & & \\ & & & 5 & 6 & 1 \\ & & & & 5 & 6 \end{bmatrix}.$$

(a) Use expansion along the first row and first column to express $\det(A_n)$ in terms of $\det(A_{n-1})$ and $\det(A_{n-2})$:

$$\det(A_n) = a \det(A_{n-1}) + b \det(A_{n-2}).$$

Use this formula to find $\det(A_4)$. ($\det(A_4)$ is a big number, close to 1000.)

(b) Use the formula from the previous part to give an exact expression for $\det(A_n)$. (If you could not determine a and b in part (a), take $a = 4$, $b = 5$.)

PROBLEM 9. Let A be a $p \times m$ matrix and B a $m \times n$ matrix.

(a) If $Rank(A) = m$ and $Rank(B) = n$, find $Rank(AB)$. (Give a complete argument. A number is not enough.)

(b) If $Rank(A) = Rank(B) = m$, find $Rank(AB)$. (Again, a complete proof is required.)

PROBLEM 10. Consider a population model with three populations X_n, Y_n, and Z_n at year n. The change in the populations is described by the model

$$\begin{aligned} X_{n+1} &= 2X_n + 1.5Y_n - 3Z_n \\ Y_{n+1} &= 6.5Y_n - 9Z_n \\ Z_{n+1} &= 3Y_n - 4Z_n \end{aligned}$$

Describe the behavior of this population model as $n \to \infty$: find all initial conditions X_0, Y_0, Z_0 such that the populations grow to infinity, and all initial conditions such that the populations die out. (Do not worry about populations being negative.)

End of exam.