Mathematics 221, Matrix Algebra, Final Exam

The University of British Columbia, 2019–2020, Winter Session, Term 1.

There are ten (10) questions, each worth 10 points. Answer all questions. The exam should be 21 pages long, and conclude with the symbol '.oOo.'

Show all your work. No notes, calculators or textbooks are permitted.

Last Name: **First Name:**

Student Number: **Signature:**

101 A. Alfieri (9:30am TuTh)	☐	104 J. Ray (1pm)	☐
102 D. Coombs (10am)	☐	106 M. Lange (3pm)	☐
103 B. Williams (1pm)	☐		

Examiners' use only

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(1) (a) Consider the linear system

$$\begin{aligned} -6x_1 + 6x_2 - 11x_3 - 3x_4 &= -12 \\ 2x_1 - 2x_2 + 2x_3 - 4x_4 &= 14 \\ x_1 - x_2 - 5x_4 &= 13 \\ 3x_1 - 3x_2 + 6x_3 + 3x_4 &= 3 \end{aligned}$$

State whether this system is consistent or inconsistent. If it is consistent, write down the set of solutions in parametric vector form. If it is inconsistent, explain how you can tell. (5pts)

(b) Consider the following linear system

$$\begin{aligned} -6x_1 + 6x_2 &= -12 \\ x_1 - x_2 &= 13 \\ ax_1 - bx_2 &= 3 \end{aligned}$$

Find values a, b, if they exist, such that the system is consistent. If such values do not exist, specify why they cannot exist. (3pts)

(c) A common mistaken belief is that a system of linear equations with more variables than equations must have infinitely many solutions. Write down a system that shows that this belief is mistaken. (2pts)

(2) In each case, either give an example or state that no such example exists (you don't have to explain why not). All parts are (2pts)

(a) A subspace W of $\mathbb{R}^3$ and two vectors $\mathbf{v}$ and $\mathbf{w}$, both in W, such that $\mathbf{v} - \mathbf{w}$ is not in W.

(b) A square matrix A such that $\det(A) \neq \det(-A)$.

(c) A matrix A and a vector $\mathbf{b}$ such that the homogeneous system $A\mathbf{x} = 0$ has a unique solution, but such that $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions.

(d) A 3×3 matrix A of real numbers such that A has no real eigenvalues.

(e) The matrix A of a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$, different from the identity, such that the two sets

$$\left\{ \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}, \quad \left\{ T\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix} \right), T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) \right\}$$

are equal.

(3) (a) Write down the formula for the inverse of an invertible 2×2 matrix: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ (2pts)

(b) Is

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 2 & 4 \\ 1 & -1 & 0 & 1 \\ 0 & 2 & 1 & 3 \end{bmatrix}$$

an invertible matrix? Give a clear justification of your answer (no credit will be given for an answer without a justification). (3pts)

(c) Let A be an $m \times m$ invertible matrix. Moreover let B and C be $n \times m$ matrices that satisfy

$$(B - C)A = 0.$$

Show that $B = C$. (2pts)

(d) Let A be an $n \times n$ matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}.$$

Is this matrix invertible? If it is, find the inverse. If it is not, explain why not. (3pts)

(4) In each of the following, indicate the correct answer (and only the correct answer) by circling the appropriate letter. You do not have to show your work in this question. All parts are (2pts).

(a) Let M be the 2×2 matrix such that $\mathbf{x} \mapsto M\mathbf{x}$ is the rotation by $90°$ counterclockwise around the origin. What are the eigenvalues of M?

A: $0, i$ **B:** i (a repeated eigenvalue) **C:** $i, -i$ **D:** $1+i, 1-i$ **E:** $\frac{i\pi}{2}, -\frac{i\pi}{2}$

(b) Suppose A is a 2×2 matrix $A = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$. What is A^{2019}?

A: $\begin{bmatrix} -1 & 2019 \\ 0 & 1 \end{bmatrix}$ **B:** $\begin{bmatrix} 2019 & 1 \\ 0 & 2019 \end{bmatrix}$ **C:** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **D:** $\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$ **E:** $\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$

(c) Consider the following statements about an $n \times n$ matrix A:
1: If the map $\mathbf{x} \to A\mathbf{x}$ is **one-to-one** then it is also **onto** $\mathbb{R}^n$.
2: If the map $\mathbf{x} \to A\mathbf{x}$ is **onto** $\mathbb{R}^n$ then it is also **one-to-one**.

Which of the statements above are true?

A: Both statements are true
B: Only statement 1 is true
C: Only statement 2 is true
D: Both statements are false

(d) A population of raccoons has two classes: juveniles and adults. Let the population in year n be given by the vector $\mathbf{x}_n = \begin{bmatrix} j_n \\ a_n \end{bmatrix}$ where j_n and a_n indicate juveniles and adults at year n, respectively.

Wildlife biologists determine that each year, 50% of the juveniles mature to become adults in the following year (the rest die), and 50% of the adults survive to the following year. Additionally, each adult has 2 offspring each year.

Suppose that the raccoon population in 2019 is given by $\mathbf{x}_{2019} = \begin{bmatrix} 12 \\ 8 \end{bmatrix}$. Find the population vector for 2017, that is, calculate $\mathbf{x}_{2017}$.

A: $\begin{bmatrix} 24 \\ 10 \end{bmatrix}$ **B:** $\begin{bmatrix} 20 \\ 17 \end{bmatrix}$ **C:** $\begin{bmatrix} 10 \\ 6 \end{bmatrix}$ **D:** $\begin{bmatrix} 20 \\ 4 \end{bmatrix}$ **E:** $\begin{bmatrix} 7 \\ 5 \end{bmatrix}$

(e) Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation such that

$$T\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad T\left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

What is $T\left(\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}\right)$?

A: $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ **B:** $\begin{bmatrix} 7 \\ 5 \end{bmatrix}$ **C:** $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ **D:** $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ **E:** It is not possible to determine.

(5) Let

$$A = \begin{bmatrix} 4 & 0 & 2 \\ 0 & 3 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

(a) Determine the eigenvalues of A. (4pts)

(b) For each eigenvalue you found, write down a basis for the associated eigenspace. (4pts) For your convenience, we reprint the matrix A on this page:

$$A = \begin{bmatrix} 4 & 0 & 2 \\ 0 & 3 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

(c) Write down an invertible matrix P so that $P^{-1}AP$ is a diagonal matrix, or state that no such P exists. (2pts)

(6) (a) Find the dimension of the subspace W in $\mathbb{R}^5$ spanned by the following three vectors (3pts)

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \\ 0 \\ 0 \end{bmatrix}.$$

(b) Find a basis of the orthogonal complement $W^{\perp}$. (3pts)

(c) Let V be a subspace of $\mathbb{R}^7$ and let $\mathbf{x}$ be a vector in $\mathbb{R}^7$. Suppose

$$\mathbf{v}_1 + \mathbf{u}_1 = \mathbf{x} = \mathbf{v}_2 + \mathbf{u}_2,$$

where $\mathbf{v}_1, \mathbf{v}_2$ are in V and $\mathbf{u}_1, \mathbf{u}_2$ are in $V^\perp$. Show that $\mathbf{v}_1 = \mathbf{v}_2$. (4pts)

(7) (a) Suppose A is an $n \times m$ matrix where $n > m$. Indicate the correct way to complete the sentence. (2pts)

The set of vectors $\mathbf{b}$ in $\mathbb{R}^n$ such that $A\mathbf{x} = \mathbf{b}$ has at least one solution is...

A: the null space of A.
B: $\mathbb{R}^m$.
C: the column space of A.
D: $\mathbb{R}^n$.

(b) The system

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 2 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 11 \\ 0 \\ 11 \\ 0 \end{bmatrix}$$

is inconsistent. Find the least-squares solution of the system $A\mathbf{x} = \mathbf{b}$. (6pts)

(c) Consider the solution you found in the previous part. Call this vector $\mathbf{v}$. What is the distance between $A\mathbf{v}$ and $\begin{bmatrix} 11 \\ 0 \\ 11 \\ 0 \end{bmatrix}$? (2pts)

(8) Let

$$W = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \right\}.$$

The following has been calculated already:

$$\text{proj}_W \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \end{bmatrix}.$$

(a) Calculate $\text{proj}_W \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\text{proj}_W \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. (3 pts)

(b) The transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by $T(\mathbf{x}) = \text{proj}_W \mathbf{x}$ is linear. Write down the matrix A of T. (2pts)

(c) Write down a basis for the null space of A. (2pts)

(d) Calculate $(I_3 - A)^{888}$. (3pts)

(9) There are two cities, Delta and Surrey. Every year

- 10% of the population of Delta migrates to Surrey,
- 20% of the population of Surrey migrates to Delta.

Assume there are no other changes to the population of Delta or Surrey. In 2019 it was estimated that the population of Delta was 100,000 people, and that of Surrey was 500,000 people. Let d_n and s_n respectively denote the population of Delta and Surrey n years from now.

(a) (2pts) Write down a matrix A such that

$$\begin{bmatrix} d_n \\ s_n \end{bmatrix} = A \begin{bmatrix} d_{n-1} \\ s_{n-1} \end{bmatrix} .$$

(b) (3pts) Let λ_1 and λ_2 denote the two eigenvalues of the matrix A. Compute λ_1 and λ_2 and two associated eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$. Find some constants a and b such that:

$$\begin{bmatrix} d_n \\ s_n \end{bmatrix} = a\lambda_1^n \mathbf{x}_1 + b\lambda_2^n \mathbf{x}_2 .$$

(c) (2pts) In the long run, what will the population of Surrey be?

(d) (3pts) Suppose now that, as well as the movement between Delta and Surrey, people immigrate from elsewhere (Abbotsford). This changes the dynamical system so that

$$\begin{bmatrix} d_n \\ s_n \end{bmatrix} = \begin{bmatrix} 1.1 & 0.2 \\ 0.1 & 1 \end{bmatrix} \begin{bmatrix} d_{n-1} \\ s_{n-1} \end{bmatrix}.$$

Assume that d_0 and s_0 are the same as in the previous case. In the long run, what will the ratio d_n/s_n be?

(10) Let A be a 3×3 matrix, and let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a linearly independent set in $\mathbb{R}^3$ such that

$$A\mathbf{v}_1 = \mathbf{v}_1, \quad A\mathbf{v}_2 = -\mathbf{v}_2, \quad A\mathbf{v}_3 = \mathbf{0}.$$

(a) Let $\mathbf{w} = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$. Show that $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are all in $\mathrm{Span}\{\mathbf{w}, A\mathbf{w}, A^2\mathbf{w}\}$. (4pts)

(b) The matrix A and the vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are the same as in the previous part. Let W be a 2-dimensional subspace of $\mathbb{R}^3$ such that, whenever a vector $\mathbf{x}$ is in W, the vector $A\mathbf{x}$ is also in W. Suppose a vector

$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

is in W where $c_1 \neq 0$ and $c_2 \neq 0$. Show that $c_3 = 0$. (6pts)

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