Final Exam

Apri 25, 2017, 12:00–14:30

No books. No notes. No calculators. No electronic devices of any kind.

Problem 1. (5 points)
Find all solutions of the equation $A\mathbf{x} = \mathbf{b}$ with

$$A = \begin{pmatrix} 1 & 1 & 2 & 3 \\ 2 & 0 & 0 & 2 \\ 3 & 2 & 4 & 7 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 1 \\ 4 \\ 4 \end{pmatrix},$$

and express them in parametric vector form.

Problem 2. (5 points)

Find the inverse of the matrix $B = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 3 & 1 & 2 \end{pmatrix}$, if it exists.

Problem 3. (3+2=5 points)

(a) Find the point on the plane W spanned by $\begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$, that is closest to the point $\mathbf{y} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$.

(b) Find the distance of $\mathbf{y}$ from W.

Problem 4. (5 points)
Find the least-squares solution to the inconsistent system of equations $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} 1 & 0 \\ -1 & 2 \\ 1 & -1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} -3 \\ -1 \\ 5 \\ 1 \end{pmatrix}.$$

Problem 5. (4+2=6 points)
Imagine you want to choose a three course meal in a restaurant and you want to spend exactly 60$ in total. You plan to dedicate 25% of the total meal price (appetizer, main course and dessert) to the tax and gratuity. In addition, you want to choose a dessert and an appetizer which in total cost twice as much as your main course. Assume that the menu can provide you with options at any price for each course.

(a) Set up a system of equations for the indeterminants a, m, d and t, for the dollar amounts to be spent on the appetizer, the main course, dessert and tax plus gratuity, respectively. Find the general solution in parametric vector form.

(b) Under the additional constraint that no menu item has a negative price, find the maximum amount of money you can spend on dessert.

Problem 6. (5 points)

Compute the determinant of the matrix $A = \begin{pmatrix} 0 & 4 & 0 & 0 & 0 \\ 2 & 1 & 0 & -2 & 0 \\ 2 & 5 & -3 & 0 & 2 \\ 3 & 2 & 3 & -1 & 3 \\ 4 & 3 & 3 & -4 & 0 \end{pmatrix}$.

Problem 7. (2+2+2+2=8 points)

(a) Let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that reflects points through the line $y = x$. Find the standard matrix of S.

(b) Let $R : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation which has eigenvalues 1 and -1, with corresponding eigenvectors $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$, respectively. Find the standard matrix of R.

(c) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation $T = S \circ R$, that is, $T(\mathbf{x}) = S\big(R(\mathbf{x})\big)$. Find the standard matrix of T.

(d) Explain why T is a rotation, and find $\tan\theta$, where θ is the (counterclockwise) rotation angle.

Problem 8. (2+2+2=6 points)

Suppose A is a 3×4 matrix, whith column vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$. Assume that the null space of A is spanned by the vector $\begin{pmatrix} 1 \\ 3 \\ 2 \\ 4 \end{pmatrix}$.

(a) Write the third column $\mathbf{v}_3$ of A as a linear combination of the other three columns.

(b) What is the rank of A?

(c) Is the linear transformation defined by A onto?

Problem 9. (2+2+2=6 points)

(a) Write down a non-zero 2×2 matrix A, satisfying $A^2 = 0$.

(b) Write down an invertible 2×2 matrix B satisfying $B^3 = -B$.

(c) Write down a 3×3 upper triangular matrix $C = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}$, with eigenvalues 2 and 3, which is not diagonalizable.

Problem 10. (3+3=6 points)

Let A be the 3×3 matrix with eigenvectors $\mathbf{v}_1 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}$, and with corresponding eigenvalues $\lambda_1 = 2$, $\lambda_2 = 1$, and $\lambda_3 = 0$, respectively. Let $\mathbf{r} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}$.

(a) Express $\mathbf{r}$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$.

(b) Find $(A^{99} - \frac{1}{3}I_3)\mathbf{r}$.

Problem 11. (5+2=7 points)
It is given that, for $n \geq 0$,

$$\begin{aligned} a_{n+1} &= -3a_n + 4b_n \\ b_{n+1} &= -6a_n + 7b_n, \end{aligned}$$

and $a_0 = 1$, $b_0 = 2$.

(a) Find explicit formulas for a_n, b_n.

(b) Find $\lim_{n\to\infty} \frac{a_n}{b_n}$.

Problem 12. (2+2=4 points)

Consider the matrix $B = \begin{pmatrix} 1 & 0 & b \\ 1 & 1 & 1 \\ -1 & 0 & 0 \end{pmatrix}$.

(a) Find all values of b, such that 3 is an eigenvalue of B.

(b) Set $b = 0$, and determine whether or not B diagonalizable. Justify your answer with appropriate facts.

Problem 13. (2+3+2=7 points)
Every year, 10% of the population of Richmond moves to Vancouver, and 20% of the population of Vancouver moves to Richmond. Assume that there are no other effects on the populations of these two cities.

(a) If the total population of the two cities is 3 million, what are the populations of the two cities in the long run?

(b) Assuming that in the current year, the population of Vancouver is 2 million, and that of Richmond is 1 million. Find precise formulas for the values of the poplulations of the two cities after n years.

(c) Continuing with the assumptions of (b), after how many years will the population of Richmond for the first time surpass the population of Vancouver?