The University of British Columbia

23 April, 2013 - Final Examination

Mathematics 221
Matrix Algebra

Closed book examination — Time: 2.5 hours

Last Name ____________________ **First** _____________ **Signature** ___________________

Student Number ____________________ **Section (circle one)** 201 202 203

Special Instructions:

No calculators, open books or notes are allowed.

There are 12 problems in this exam. Each problem is worth 10 marks.

Rules governing examinations

- Each candidate must be prepared to produce, upon request, a UBC-card for identification.
- Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination.
- Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.
  (a) Having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners.
  (b) Speaking or communicating with other candidates.
  (c) Purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received.
- Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.
- Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

1		10
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
Total		120

PROBLEM 1. Consider the following system of linear equations:

$$\begin{array}{rcrcrcl} 3x & + & 4y & + & 7z & = & 3 \\ -9x & + & 6y & + & 9z & = & 3 \\ 45x & - & 12y & + & hz & = & k \end{array}$$

For which values of h and k does the system have:
a) no solutions?
b) exactly one solution?
c) infinitely many solutions?

PROBLEM 2. Let

$$A = \begin{bmatrix} 1 & 2 & 1 & 3 & 2 \\ 3 & 4 & 9 & 0 & 7 \\ 2 & 3 & 5 & 1 & 8 \\ 2 & 2 & 8 & -3 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 1 & 3 & 2 \\ 0 & 1 & -3 & 5 & -4 \\ 0 & 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrices A and B are related by $B = MA$, where M is an invertible 4×4 matrix.

a) Find the rank and a basis of the row space of A.

b) Find the dimension and a basis of the nullspace of A.

c) Find a basis of the column space of A.

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Problem 2 (continued). Recall that

$$A = \begin{bmatrix} 1 & 2 & 1 & 3 & 2 \\ 3 & 4 & 9 & 0 & 7 \\ 2 & 3 & 5 & 1 & 8 \\ 2 & 2 & 8 & -3 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 1 & 3 & 2 \\ 0 & 1 & -3 & 5 & -4 \\ 0 & 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and that the matrices A and B are related by $B = MA$, where M is an invertible 4×4 matrix.

d) Find the general solution of the system of equations:

$$\begin{array}{rcrcrcrcl} x & + & 2y & + & z & + & 3w & = & 2 \\ 3x & + & 4y & + & 9z & & & = & 7 \\ 2x & + & 3y & + & 5z & + & w & = & 8 \\ 2x & + & 2y & + & 8z & - & 3w & = & 5 \end{array}$$

Is there a solution in which $z = 0$? Is so, find it. Is there a solution in which $w = 0$? If so, find it.

PROBLEM 3. There are two linear transformations of $\mathbb{R}^2$ that map the square with vertices at $(0,0),(1,0),(0,1)$, and $(1,1)$ to the square with vertices at $(0,0),(1,1),(-1,1)$, and $(0,2)$. Call them T_1 and T_2. Determine the matrices of T_1 and T_2.

PROBLEM 4. A refining company must produce 70 oz. of gold and 500 oz. of silver from ore. The ores available for purchase have the following characteristics.

Ore type	Cost	Gold yield	Silver yield
A	\$300/ton	1 oz./ton	20 oz./ton
B	\$200/ton	1 oz./ton	10 oz./ton
C	\$400/ton	2 oz./ton	10 oz./ton

How many tons of each type of ore should be purchased in order to produce the metals at least cost?

PROBLEM 5. Solve for the matrix X if $A + (BX)^T = C$, where

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -5 & 1 \\ -3 & 7 & -1 \\ 1 & -1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & 9 & 0 \\ 3 & 7 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

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PROBLEM 6. Find the determinant of

$$\begin{bmatrix} 1 & 2 & 2 & -1 & 3 \\ 2 & 6 & 3 & -3 & 7 \\ -3 & -4 & -3 & 2 & -8 \\ -2 & -6 & -2 & 2 & -7 \\ 1 & -2 & 7 & 3 & 3 \end{bmatrix}.$$

PROBLEM 7. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that reflects points through the line $3x = 4y$.

a) Find the eigenvalues and eigenvectors of the standard matrix A of T. (Note: In order to do this, you do not need to evaluate A.)

b) Using the result of a), find A.

c) Compute A^{1995}.

PROBLEM 8. Let

$$A = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 3 & 0 \\ 4 & 0 & 1 \end{bmatrix}$$

Find a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$. Do not compute the matrix P^{-1}. Show that A^{-k} approaches the zero matrix as k becomes very large.

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PROBLEM 9. Given the discrete dynamical system $\vec{x}_{n+1} = A\vec{x}_n$ with

$$A = \begin{bmatrix} 0.8 & 0.5 \\ 0.2 & 0.5 \end{bmatrix}$$

Find the eigenvalues and eigenvectors of the system. Then find closed formulae for the components of $\vec{x}_n$ with the initial value $\vec{x}_0 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$.

PROBLEM 10. Let $W = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix} \right\}$. Find a basis for the orthogonal complement $W^{\perp}$ of W.

PROBLEM 11. Consider the following matrix A and vector $\vec{b}$:

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 2 \end{bmatrix}.$$

Find the least squares solution $\vec{x}^*$ for the linear system $A\vec{x} = \vec{b}$. Use this to find a linear function whose graph best fits the points $(-1, 0)$, $(0, 1)$, $(1, 2)$, and $(2, 2)$ in $\mathbb{R}^2$.

PROBLEM 12. Consider the plane $x - y + z = 0$.

a) Find the 3×3 matrix P_1 which represents projection of $\mathbb{R}^3$ onto a vector orthogonal to this plane.

b) Let $\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Find vectors $\vec{v}$ and $\vec{w}$ such that $\vec{u} = \vec{v} + \vec{w}$, where $\vec{v}$ is in the plane and $\vec{w}$ is perpendicular to the plane.

c) Find the 3×3 matrix P_2 which represents the projection of $\mathbb{R}^3$ onto the given plane.

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