The University of British Columbia

Final Examination - December 18, 2009

Mathematics 221

Circle one:

Section 101	Section 102	Section 103
MWF 8-9	MWF 10-11	MWF 1-2

Closed book examination Time: 2.5 hours

Last Name ____________ **First** ____________ **Signature** ____________

Student Number ____________

Special Instructions:

No notes or calculators are allowed. Answer all 12 questions on the sheets provided - use the backs of the sheets and blank sheets at the end of the test if necessary.

Rules governing examinations

- Each candidate must be prepared to produce, upon request, a UBCcard for identification.
- Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination.
- Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.

  (a) Having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners.

  (b) Speaking or communicating with other candidates.

  (c) Purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received.
- Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.
- Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

1		10
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
Total		120

PROBLEM 1. Consider the system of equations:

$$\begin{array}{rcrcrcl} x_1 & + & x_2 & - & x_3 & = & 2 \\ x_1 & + & 2x_2 & + & x_3 & = & 3 \\ x_1 & + & x_2 & + & (c^2-5)x_3 & = & c \end{array}$$

Find all values of c such that the system has:

a. no solutions

b. a unique solution

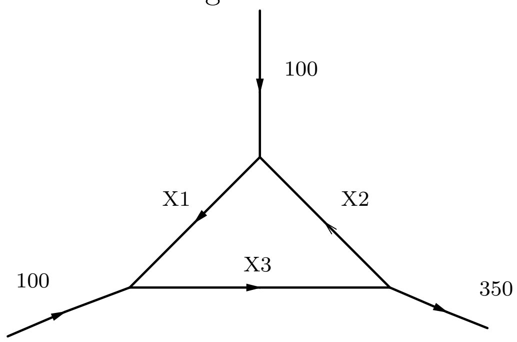
c. infinitely many solutions

In case **c.** write the general solution in the parametric vector form.

PROBLEM 2. Find the determinants of the matrices

$$a.\ \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}, \qquad b.\ \begin{bmatrix} 1 & 0 & 3 & 0 \\ 2 & 1 & 4 & -1 \\ 3 & 2 & 4 & 0 \\ 0 & 3 & -1 & 1 \end{bmatrix}.$$

PROBLEM 3. Consider the traffic flow diagram:



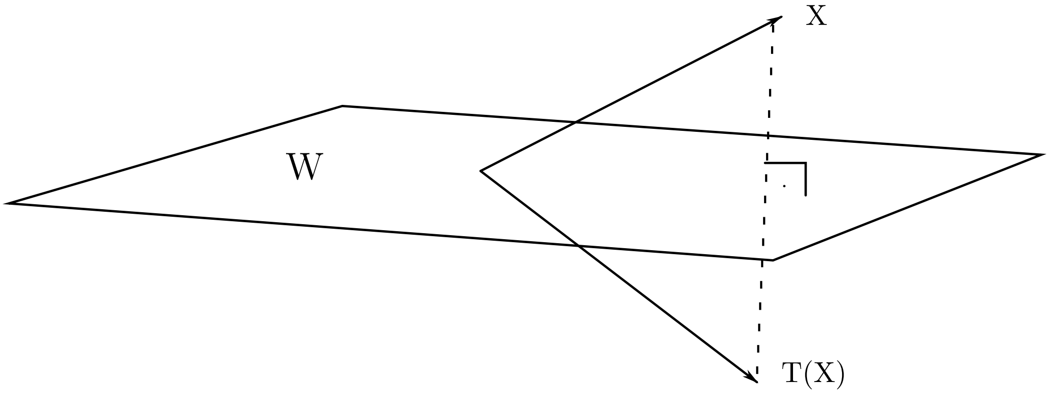
a. Find a system of 3 equations in 3 variables that describes this model.

b. The system is clearly inconsistent because the total infolw does not equal total outflow. Find all least squares solutions to the system.

PROBLEM 4. Let W be the subspace of $\mathbb{R}^3$ spanned by $\vec{w}_1$ and $\vec{w}_2$, where

$$\vec{w}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \vec{w}_2 = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}.$$

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the reflection across W. Find the standard matrix of T.



PROBLEM 5. Let

$$W = \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 4 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ -6 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 8 \\ 2 \end{bmatrix} \right\}.$$

a. Find a basis for W.

b. Find a basis for $W^{\perp}$.

c. Find all c such that $[c, 1, 1, 0]^T$ lies in W.

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PROBLEM 6. Let

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}.$$

Find an invertible matrix P and a diagonal matrix D, such that $A = PDP^{-1}$. (No need to find P^{-1}.)

PROBLEM 7. Two drink makers Hard and Soft compete for customers. Each year 20% of Hard's customers shift to Soft and 30% of Soft's customers shift to Hard. Assume that this year the drink makers have equal market share.

a. What will be the market distribution next year?

b. What will be the market distribution in the distant future?

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PROBLEM 8. A discrete dynamical system is described by:

$$x_{n+1} = 17x_n + 9y_n,$$
$$y_{n+1} = -30x_n - 16y_n.$$

Given that $x_0 = 1, y_0 = -1$, find x_{30}, y_{30}.

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PROBLEM 9. Find the least squares fit of a parabola $y = a + bx + cx^2$ to the data (x_i, y_i): $(0, 1), (1, 0), (2, 5), (3, 6)$.

PROBLEM 10. Find the numbers a, b, c which make the matrices below diagonalizable. (No need to diagonalize them.)

$$a. \begin{bmatrix} 0 & 1 & 2 \\ 0 & 3 & a \\ 0 & 0 & 0 \end{bmatrix}, \qquad b. \begin{bmatrix} 2 & 1 & b & 3 \\ 0 & 3 & -1 & c \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

PROBLEM 11. Let

$$A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 3 & 2 \\ -1 & 2 & -2 \\ 2 & -1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 7 & -3 & 3 \\ -2 & 16 & -2 \\ 6 & -1 & 10 \end{bmatrix}.$$

Find a matrix X satisfying the equation

$$A^T AX + 2X = BX + B^T.$$

PROBLEM 12. Mark each statement either True or False. You do **not** have to justify your answer.

- **a.** A system of 4 linear equations in 3 variables is always inconsistent.
- **b.** If A is a 4×3 matrix, then there exists a vector $\vec{b}$ such that $A\vec{x} = \vec{b}$ has no solutions.
- **c.** If the matrix A has 6 rows and 9 columns, then $\dim(Nul(A)) \geq 3$.
- **d.** If a 5×6 matrix A has rank 4, then $\dim(Nul(A)) = 1$.
- **e.** If $T : \mathbb{R}^n \to \mathbb{R}^n$ is the orthogonal projection onto a subspace W, then the standard matrix of T is diagonalizable.
- **f.** The rank of any upper-triangular $n \times n$ matrix is the number of nonzero entries on its diagonal.
- **g.** If $\vec{v}_1, \ldots, \vec{v}_n$ span $\mathbb{R}^5$, then n must be equal to 5.
- **h.** If $\vec{v}_1, \ldots, \vec{v}_m$ are linearly independent vectors in $\mathbb{R}^n$, then they form a basis of a subspace W of $\mathbb{R}^n$.
- **i.** If the system $A^2\vec{x} = \vec{b}$ is consistent, then $A\vec{x} = \vec{b}$ must also be consistent.
- **j.** The matrix

$$A = \begin{bmatrix} -0.6 & -0.8 \\ -0.8 & 0.6 \end{bmatrix}$$

is the matrix of rotation by some angle θ in $\mathbb{R}^2$.

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