Final Exam

April 22, 2006

No books. No notes. No calculators.
Time: 15:30 - 18:00, which is 150 minutes.

Problem 1. (8 points)
Find the general solution in parametric vector form of the inhomogeneous system of linear equations

$$\begin{aligned} x_3 + x_4 &= 7 \\ x_2 + x_3 &= 5 \\ x_1 + x_2 &= 3 \\ x_1 + x_4 &= 5 \end{aligned}$$

Problem 2. (8 points)
Find the inverse of the matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$

Problem 3. (10 points)
Consider the matrix

$$A = \begin{pmatrix} 0 & 2 & 4 & -2 \\ 1 & -1 & -1 & 2 \\ 0 & -2 & -4 & 2 \\ 1 & -1 & -1 & 2 \end{pmatrix}$$

(a) Find a basis for the column space of A.
(b) Find a basis for the nullspace of A.

Problem 4. (5 points)
(a) What is the rank of a 4×5 matrix whose null space is three dimensional?
(b) What is the rank of a 4×4 matrix with determinant -1?

Problem 5. (6 points)

The colour of light can be represented in a vector $\begin{pmatrix} R \\ G \\ B \end{pmatrix}$, where $R =$ amount of red, $G =$ amount of green and $B =$ amount of blue. The human eye and the brain transform the incoming signal into the signal $\begin{pmatrix} I \\ L \\ S \end{pmatrix}$, where

$$\begin{aligned} I &= \text{intensity} = \frac{R+G+B}{3} \\ L &= \text{long-wave signal} = R - G \\ S &= \text{short-wave signal} = B - \frac{R+G}{2} \end{aligned}$$

(a) Find the matrix P representing the transformation from $\begin{pmatrix} R \\ G \\ B \end{pmatrix}$ to $\begin{pmatrix} I \\ L \\ S \end{pmatrix}$.

(b) Consider a pair of yellow sunglasses for water sports which cuts out all blue light and passes all read and green light. Find the matrix A which represents the transformation incoming light undergoes as it passes through the sunglasses.

(c) Find the matrix for the composite transformation which light undergoes as it first passes through the sunglasses and then the eye.

Problem 6. (6 points)
Find the determinant of the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1+a & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+c & 1 \end{pmatrix}$$

Problem 7. (10 points)
Consider the system of differential equations

$$\vec{x}\,' = A\vec{x} \qquad \text{where} \qquad A = \begin{pmatrix} 0 & 1 \\ 1/4 & 3/4 \end{pmatrix}$$

(a) Classify the origin as either an attractor, repellor, or saddle point. Justify your answer.

(b) Solve the initial value problem $\vec{x}\,' = A\vec{x}$, with $\vec{x}(0) = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$.

Problem 8. (6 points)
Determine whether or not the following matrix is diagonalizable. Show all work. Do **not** diagonalize the matrix.

$$A = \begin{pmatrix} 0 & 2 & 4 \\ 0 & -1 & -2 \\ 1 & 0 & -1 \end{pmatrix}$$

Problem 9. (8 points)
The following matrix A is diagonalizable.

$$A = \begin{pmatrix} 3 & -2 & 0 \\ -\frac{1}{2} & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

(a) Find all eigenvalues of A and their (algebraic) multiplicities.
(b) For each eigenvalue find a basis of the corresponding eigenspace.

Problem 10. (6 points)
Suppose you know the following about a 2×2-matrix A.

$$A \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \qquad A \begin{pmatrix} 1 \\ 4 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

(a) Find A. (Hint: Write A as $A = PDP^{-1}$, for a diagonal matrix D.)
(b) Find $\lim_{n\to\infty} A^n$.

Problem 11. (4 points)
Find the matrix of the quadratic form Q in four variables given by the formula

$$Q(x_1, x_2, x_3, x_4) = \det \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$$

Problem 12. (8 points)
Let $\vec{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and consider the quadratic form Q in two variables defined as follows: for a vector $\vec{x}$ in $\mathbb{R}^2$,

$$Q(\vec{x}) = (\vec{x} \cdot \vec{v})^2$$

(a) Compute the matrix A of Q.
(b) Perform an orthogonal change of variables $\vec{x} = P\vec{y}$, that transforms Q into a quadratic form without cross-product term. Give P and the new quadratic form.
(c) Give a geometric description of the set of all vectors $\vec{x}$ for which $Q(\vec{x}) = 0$ (i.e. is it a point, a single line, a union of lines, a hyperbola, or an ellipse?). Justify your answer and **draw a sketch**.

Problem 13. (6 points)
Let E be the plane in $\mathbb{R}^3$ spanned by the orthogonal vectors

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \qquad \text{and} \qquad \vec{v}_2 = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$$

The reflection across E is the linear transformation $R\colon \mathbb{R}^3 \to \mathbb{R}^3$ defined by the formula

$$R(\vec{x}) = 2\,\mathrm{proj}_E(\vec{x}) - \vec{x}$$

(a) Compute $R(\vec{x})$ for

$$\vec{x} = \begin{pmatrix} 12 \\ 6 \\ 0 \end{pmatrix}$$

(b) Find the eigenspace of R corresponding to the eigenvalue 1. That is, find the set of all vectors $\vec{x}$ for which $R(\vec{x}) = \vec{x}$. Justify your answer.

Problem 14. (6 points)
Suppose E is the column space of the matrix

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}$$

Then E is a plane in $\mathbb{R}^3$. Determine the orthogonal projection of

$$\vec{b} = \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix}$$

onto E. (Hint: The columns of A are *not* orthogonal! Use the fact that if $\hat{x}$ is any least square solution of $A\vec{x} = \vec{b}$, then $A\hat{x} = \mathrm{proj}_E(\vec{b})$.)

Problem 15. (4 points)
Suppose A is a 3×3 matrix such that $A^3 = 0$.
(a) Explain why A cannot be invertible.
(b) Show that $(I - A + A^2)$ is invertible.