Math 221 Final Exam (April 2005)

Last Name: ______________________________ **First name**: ______________________________

Student #: ______________________________ **Signature**: ______________________________

Circle your section #: **201 (Culibrk), 202 (Pakzad), 203 (Li)**

I have read and understood the instructions below:

Please sign:

Instructions:

1. No notes or books are to be used in this exam. No calculators are allowed.
2. There are 11 pages (including this cover page) in the exam. Justify every answer, and show your work. Unsupported answers will receive no credit.
3. You will be given 2.5 hrs to write this exam. Read over the exam before you begin. You are asked to stay in your seat during the last 5 minutes of the exam, until all exams are collected.
4. At the end of the hour you will be given the instruction "Put away all writing implements and remain seated." *Continuing to write after this instruction will be considered as cheating.*
5. **Academic dishonesty:** Exposing your paper to another student, copying material from another student, or representing your work as that of another student constitutes academic dishonesty. Cases of academic dishonesty may lead to a zero grade in the exam, a zero grade in the course, and other measures, such as suspension from this university.

Question	grade	value
1		10
2		12
3		12
4		12
5		20
6		18
7		16
Total		**100**

Question 1: **[10 marks]**

A linear transformation $T\colon\ \mathbb{R}^2 \to \mathbb{R}^2$ is defined as the reflection in the line $x_1 = x_2$ followed by a counter-clockwise rotation of 90 degrees. Find the transformation matrix A such that $T(\vec{x}) = A\vec{x}$ in standard basis.

Question 2: **[12 marks]**

Are the following statements True or False? Present a brief reason or calculation to support your answer.

(a) The quadratic form $q(x_1, x_2) = 2x_1^2 + 6x_1x_2 - 6x_2^2$ is negative definite (use the method learned in matrix algebra in your answer).

(b) For $A = \begin{bmatrix} 3 & -5 \\ 2 & -3 \end{bmatrix}$, there exists a real-valued matrix S such that $S^{-1}AS = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, where λ_1 and λ_2 are the two eigenvalues of the matrix A.

Question 3: **[12 marks]**

(a) Check each of the following functions to determine if it is a linear transformation.

(b) For each linear transformation, determine if it is invertible.

$$T_1\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_1 + 2x_2 - 4x_3 \\ x_1 - 2x_3 \\ -2x_1 + 3x_2 + 3x_3 \end{bmatrix}, \quad T_2\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4x_1^2 + 5x_2 \\ 7x_1x_2 - 3x_3 \\ x_1 + x_2 + x_3 \end{bmatrix}, \quad T_3\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_3 \\ 3x_1 - 3x_3 \\ 4x_1 - 2x_2 + 7x_3 \end{bmatrix}.$$

Question 4: **[12 marks]**

Determine whether the matrix $A = \begin{bmatrix} 3 & -4 & 0 \\ 2 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is diagonalizable.

Question 5: **[20 marks]**

Consider the following system of linear equations

$$\begin{array}{rrrrcl} x_1 & & +2x_3 & +x_4 & = & 1 \\ -x_1 & +x_2 & & +x_4 & = & 1 \\ & +2x_2 & +4x_3 & +4x_4 & = & 4 \\ 2x_1 & -3x_2 & -2x_3 & -4x_4 & = & \alpha \end{array}$$

(a) For what value of α has the system at least one solution?

(b) For α found in (a), solve this system. (Express the solution in parametric vector form.)

(c) Find a basis of im(A) for the coefficient matrix

$$A = \begin{bmatrix} 1 & 0 & 2 & 1 \\ -1 & 1 & 0 & 1 \\ 0 & 2 & 4 & 4 \\ 2 & -3 & -2 & -4 \end{bmatrix}.$$

(d) Find the dimension of ker(A), i.e. find $m = dim(ker(A))$. After you found the value of m, find m linearly independent vectors that are solutions of $A\vec{x} = \vec{0}$. Do they form a basis of ker(A)?

(*Space for working on the problems in Question 5.*)

Question 6: **[18 marks]**

(a) Find a vector $\vec{v}_3$ in $\mathbb{R}^3$ that is orthogonal to both $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$.

(b) Apply the Gram-Schmidt process to $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$ to produce an orthonormal basis of $\mathbb{R}^3$: $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$.

(c) Determine if the vector $\vec{x} = \begin{bmatrix} 9 \\ -2 \\ 18 \end{bmatrix}$ is in the plane span by $\vec{u}_1$ and $\vec{u}_2$, $V = span(\vec{u}_1, \vec{u}_2)$.

(*Space for working on the problems in Question 6.*)

Question 7: **[16 marks]**

The following discrete dynamical system describes the yearly migration of wild horse populations among three areas R, G, and B. Let $r(t)$, $g(t)$, and $b(t)$ be the sizes of the horse population in areas R, G, and B respectively at the t^{th} year.

$$\vec{x}(t+1) = \begin{bmatrix} r(t+1) \\ g(t+1) \\ b(t+1) \end{bmatrix} = \begin{bmatrix} r(t)/2 + g(t)/3 + b(t)/3 \\ r(t)/2 + g(t)/3 + b(t)/2 \\ g(t)/3 + b(t)/6 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/3 & 1/3 \\ 1/2 & 1/3 & 1/2 \\ 0 & 1/3 & 1/6 \end{bmatrix} \begin{bmatrix} r(t) \\ g(t) \\ b(t) \end{bmatrix} = A\vec{x}(t),$$

where the matrix A describes how the horses move between these areas from one year to the next. The 1st column indicates that each year 1/2 of the horses in area R remain in area R and 1/2 will migrate to area G. The 2nd column shows that horses in area G will be evenly distributed in the three areas one year later. The 3rd column implies that, of the horses in area B, 1/3 will migrate to area R, 1/2 will migrate to area G, and only 1/6 will remain in area B.

We assume that no horses are lost and no new horses are added. Thus, the sum of the entries in each column is equal to 1. Assume that initially (i.e. at $t = 0$), there are a total of 350 horses all located in area B. Thus, $\vec{x}(0) = [0 \ \ 0 \ \ 350]^T$.

(a) Show that $\lambda_3 = 1$ is an eigenvalue of A. Then, use the equalities $tr(A) = \lambda_1 + \lambda_2 + \lambda_3$ and $det(A) = \lambda_1\lambda_2\lambda_3$ to find the other two eigenvalues λ_1 and λ_2.

(b) Find a vector $\vec{v}$ such that $A\vec{v} = \vec{v}$, and thus $A^t\vec{v} = \vec{v}$ for all $t > 1$.
(Such a vector is called an *equilibrium point* of A. It represents a special distribution of the horses that remains unchanged, although horses continue to move year after year following the rules set in A).

(c) Find the matrix S such that

$$S^{-1}AS = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

where λ_1, λ_2, and λ_3 are the eigenvalues found in (a).

(d) Solve $\vec{x}(t) = A^t\vec{x}(0)$ explicitly in terms of λ_1^t, λ_2^t, λ_3^t, and the eigenvectors corresponding to λ_1, λ_2, and λ_3. Then, calculate $\vec{x}(\infty) = \lim_{t\to\infty} \vec{x}(t)$.

(*Space for working on the problems in Question 7.*)