

The University of British Columbia

Final Examination - April 12th, 2016

Mathematics 215/255

Time: 2 hours

Last Name \_\_\_\_\_ First \_\_\_\_\_ Signature \_\_\_\_\_

Student Number \_\_\_\_\_

Section (circle one) Coombs (215:201) / Herrera (215:202) / Rahmani (255:201)

Special Instructions:

- Books, notes, cellphones and calculators are **not** allowed.
- Use the reverse side of each page if you need extra space.
- **Show your work.** A correct answer without intermediate steps will receive no credit.
- This exam has 12 pages, including this cover and a table of Laplace transforms.

Rules governing examinations

- Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
- Candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no candidate shall be permitted to enter the examination room once the examination has begun.
- Candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
- Candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
  - (a) speaking or communicating with other candidates, unless otherwise authorized;
  - (b) purposely exposing written papers to the view of other candidates or imaging devices;
  - (c) purposely viewing the written papers of other candidates;
  - (d) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
  - (e) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)–(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
- Candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
- Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
- Candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

1		10
2		10
3		15
4		10
5		10
6		5
7		20
8		20
Total		100

**Problem 1** (10 Marks) Your lab partner leaves a drop of bleach on the lab bench, which takes the shape of a hemisphere. At  $t = 0$  the drop has a radius of 6mm. The drop loses volume due to evaporation at a rate proportional to its surface area. When  $t = 10$  minutes, the radius is 4mm. At what time has the drop completely evaporated? Show your work.  
Note: Only the part of the drop surface that is exposed to the air can evaporate.

The drop is gone when  $t =$                       minutes.

**Problem 2** (10 Marks) a. Solve the initial value problem for  $y(t)$ . Show your work.

$$y'' + \pi^2 y = 0, \quad y(0) = 1; \quad y'(0) = 1.$$

$y(t) =$

b. Draw a clear graph of the solution, indicating the period and amplitude. Show your work.

**Problem 3** (15 Marks) Find a fundamental matrix  $\mathbf{X}(t)$  and determine the general solution to

$$\vec{x}'(t) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \vec{x} + \begin{pmatrix} -1 \\ -e^{-t} \end{pmatrix}.$$

Show your work.

$\mathbf{X}(t) =$

$\vec{x}(t) =$

**Problem 4** (10 Marks) Find the inverse Laplace transforms of the indicated functions. Show your work.

(1)

$$F(s) = \frac{2}{s^2 + 3s - 4}$$

(2)

$$G(s) = \frac{2s + 1}{s^2 - 2s + 2}$$

$f(t) =$

$g(t) =$

**Problem 5** (10 Marks) Solve the initial value problem using Laplace transform. Answers obtained using any other method will not receive any points. Show your work.

$$y'' - 2y' + 2y = \cos(t) + 2\sin(t), \quad y(0) = 2, \quad y'(0) = 0$$

$$y(t) =$$

**Problem 6** (5 Marks) For what values of  $\alpha$  does the solution to the following problem remain bounded as  $t \rightarrow \infty$ ? Hint: use your work from the previous problem. Show your work.

$$y'' - 2y' + 2y = \cos(t) + 2\sin(t), \quad y(0) = \alpha, \quad y'(0) = 0$$

The solution remains bounded for

**Problem 7** (20 Marks) A stylized model of competing species with population densities  $x(t)$  and  $y(t)$  is given by

$$\begin{aligned}\frac{dx}{dt} &= x(1 - x - y), \\ \frac{dy}{dt} &= y\left(\frac{1}{2} - \frac{y}{4} - \frac{3x}{4}\right).\end{aligned}$$

This model has critical points at  $(0, 0)$ ,  $(0, 2)$ ,  $(1, 0)$  and  $(\frac{1}{2}, \frac{1}{2})$ .

(1) Find the Jacobian matrix for this system.

$J(t) =$
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(2) Classify each critical point. Show your work.

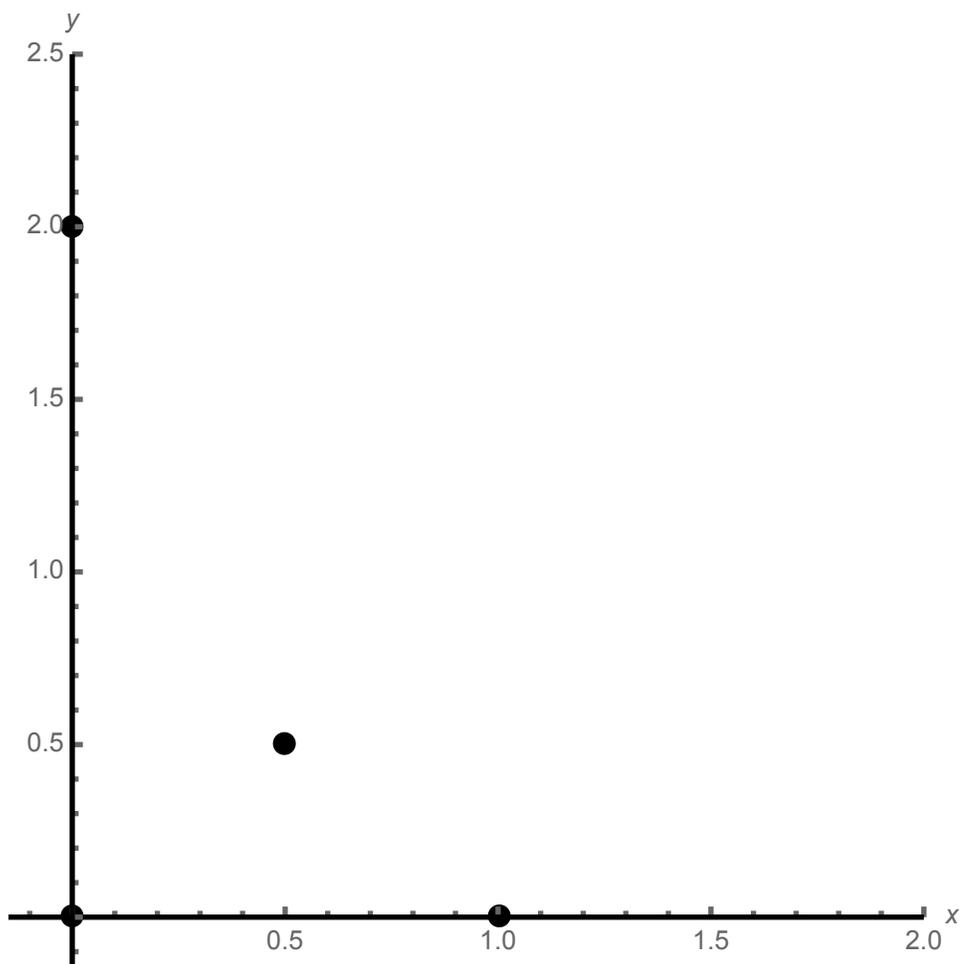
$(x, y) = (0, 0)$ is
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$(x, y) = (0, 2)$ is
----------------------

$(x, y) = (1, 0)$ is
----------------------

$(x, y) = (\frac{1}{2}, \frac{1}{2})$ is
--

(3) Sketch representative solutions of the nonlinear system on the axes below:



**Problem 8** (20 Marks) Consider a forced second order equation modeling the charge stored in a capacitor, which can be described by:

$$Q''(t) + RQ'(t) + \frac{1}{C}Q = F \cos(\omega t), \quad Q(0) = 0, \quad Q'(0) = 1,$$

$R$  and  $C$  are parameters and the forcing has amplitude  $F$  and frequency  $\omega$ .

On the following page you will find plots of different solutions  $Q(t)$  for various values of  $R$ ,  $C$ ,  $F$ , and  $\omega$ , corresponding in no particular order to:

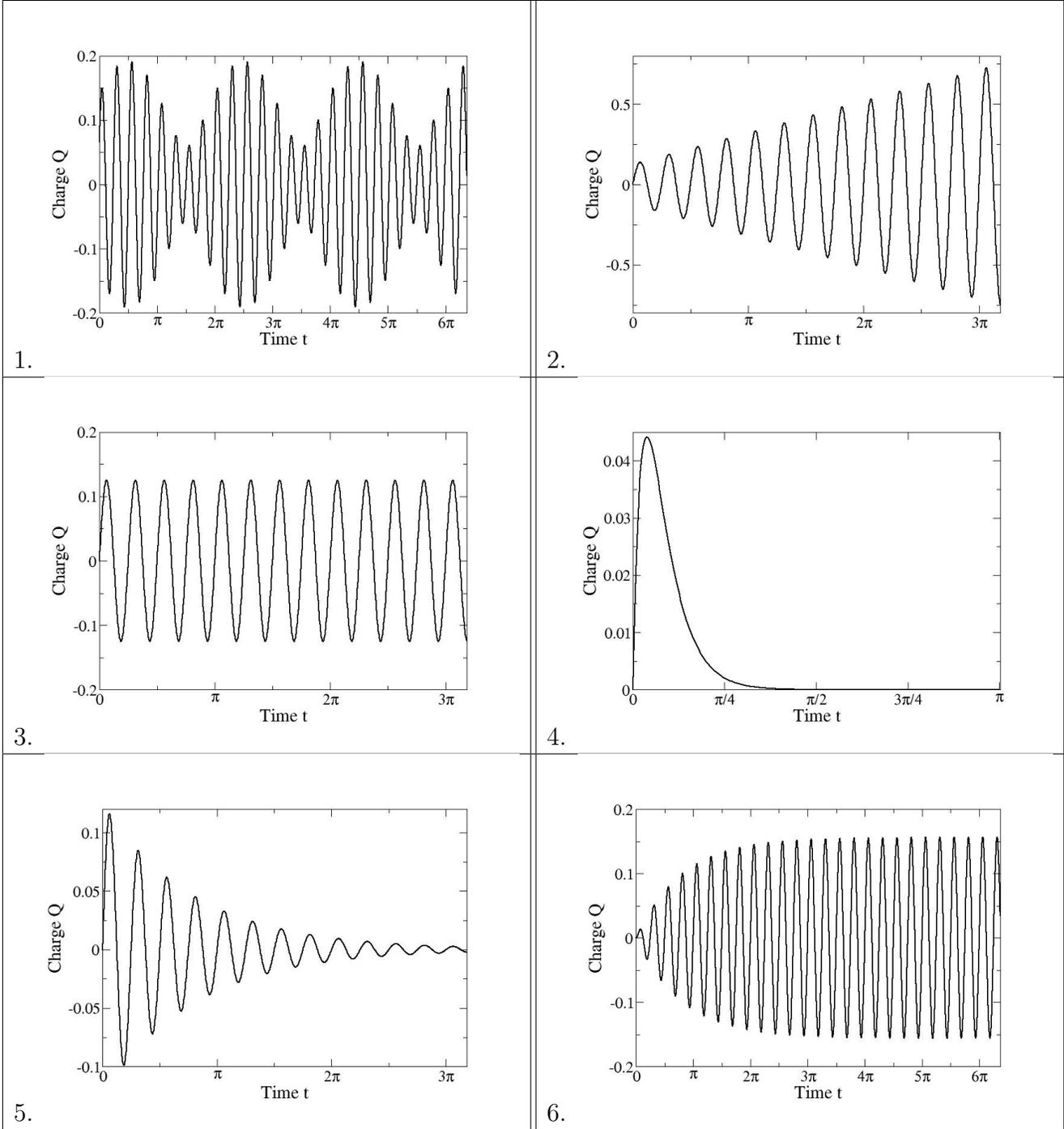
- A.  $Q''(t) + 64Q(t) = \cos(8t)$ .
- B.  $Q''(t) + 0.8Q'(t) + 64Q(t) = 0$ .
- C.  $Q''(t) + 0.8Q'(t) + 64Q(t) = \cos(8t)$ .
- D.  $Q''(t) + 17Q'(t) + 64Q(t) = 0$ .
- E.  $Q''(t) + 64Q(t) = 0$ .
- F.  $Q''(t) + 64Q(t) = \cos(7t)$ .

### COMPLETE THIS TABLE:

Equation	Graph
A	
B	
C	
D	
E	
F	

**Hints:**

1. You don't need to solve each problem completely.
2. Look at the plots carefully and don't forget to check the axis scales.
3. No work need be shown.



END OF EXAM

## Table of Laplace Transforms

$f(t)$	$\mathcal{L}[f(t)] = F(s)$	$f(t)$	$\mathcal{L}[f(t)] = F(s)$
1	$\frac{1}{s}$ (1)	$te^{at}$	$\frac{1}{(s-a)^2}$ (13)
$e^{at}f(t)$	$F(s-a)$ (2)	$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$ (14)
$\mathcal{U}(t-a)$	$\frac{e^{-as}}{s}$ (3)	$e^{at} \sin kt$	$\frac{k}{(s-a)^2 + k^2}$ (15)
$f(t-a)\mathcal{U}(t-a)$	$e^{-as}F(s)$ (4)	$e^{at} \cos kt$	$\frac{s-a}{(s-a)^2 + k^2}$ (16)
$t^n f(t)$	$(-1)^n \frac{d^n F(s)}{ds^n}$ (5)	$e^{at} \sinh kt$	$\frac{k}{(s-a)^2 - k^2}$ (17)
$f'(t)$	$sF(s) - f(0)$ (6)	$e^{at} \cosh kt$	$\frac{s-a}{(s-a)^2 - k^2}$ (18)
$f^{(n)}(t)$	$s^n F(s) - s^{n-1}f(0) - \dots - f^{(n-1)}(0)$ (7)	$t \sin kt$	$\frac{2ks}{(s^2 + k^2)^2}$ (19)
$\int_0^t f(x)g(t-x)dx$	$F(s)G(s)$ (8)	$t \cos kt$	$\frac{s^2 - k^2}{(s^2 + k^2)^2}$ (20)
$t^n$ ( $n = 0, 1, 2, \dots$ )	$\frac{n!}{s^{n+1}}$ (9)	$t \sinh kt$	$\frac{2ks}{(s^2 - k^2)^2}$ (21)
$\sin kt$	$\frac{k}{s^2 + k^2}$ (10)	$t \cosh kt$	$\frac{s^2 - k^2}{(s^2 - k^2)^2}$ (22)
$\cos kt$	$\frac{s}{s^2 + k^2}$ (11)		
$e^{at}$	$\frac{1}{s-a}$ (12)		

## Trig identities

$$\begin{aligned} \sin(A \pm B) &= \sin A \cos B \pm \sin B \cos A \\ \cos(A \pm B) &= \cos A \cos B \mp \sin A \sin B \end{aligned}$$

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