

The University of British Columbia

Final Examination - April 16th, 2014

Mathematics 215

Time: 2.5 hours

Last Name _____ First _____ Signature _____

Student Number _____ Section Lindstrom (201) / Coombs (202)

Special Instructions:

- Books, notes, cellphones and calculators are **not** allowed.
- Use the reverse side of each page if you need extra space.
- **Show your work.** A correct answer without intermediate steps will receive no credit.
- This exam has 12 pages, including this cover and a table of Laplace transforms.

Rules governing examinations

- Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
- Candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no candidate shall be permitted to enter the examination room once the examination has begun.
- Candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
- Candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
 - (a) speaking or communicating with other candidates, unless otherwise authorized;
 - (b) purposely exposing written papers to the view of other candidates or imaging devices;
 - (c) purposely viewing the written papers of other candidates;
 - (d) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
 - (e) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)–(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
- Candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
- Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
- Candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

1		63
2		13
3		12
4		12
Total		100

Problem 1

(63 Marks)

This problem has nine parts. Each part is worth 7 marks. Write your final answers in the boxes. Show your work.

A. Solve $y' + y = t$ for $y(t)$, subject to $y(0) = -2$.

$$y(t) =$$

B. Solve $\frac{dy}{dx} = xy + x + y + 1$ for $y(x)$, subject to $y(1) = 1$. Hint: Factor it.

$$y(x) =$$

C. Consider $\frac{dy}{dt} = y(1 - y)(2 - y)$. For what possible initial conditions $y(0)$ does $\lim_{t \rightarrow \infty} y(t) = 1$?

D. Find the general solution of $y'' + 16y = \sin 4t$.

$y(t) =$

E. Find the general solution of the differential equation system:

$$\frac{d\vec{y}(t)}{dt} = \begin{pmatrix} 7 & 3 \\ 5 & -7 \end{pmatrix} \vec{y}(t).$$

$$\vec{y}(t) =$$

F. Find the inverse Laplace transform of $F(s) = \frac{s^2}{(s^2+100)^2}$.

$$f(t) =$$

G. Use two steps of Euler's method with time step $h = 0.5$ to estimate $y(1)$ for the following problem:

$$y'(t) = (2t - y)^2, \quad y(0) = 2$$

$$y(1) \simeq$$

H. The angle of a pendulum satisfies

$$L\theta''(t) + c\theta'(t) + g\theta(t) = 0$$

where L is the length (which can be changed), $c = 0.1$ m/s, and $g = 10$ m/s². Find the maximum frequency (in Hertz) of the pendulum over all possible lengths L .

Maximum frequency =

I. Given $L[\sqrt{t}] = \frac{\sqrt{\pi}}{2s^{3/2}}$, find $L\left[\frac{1}{\sqrt{t}}\right]$.

$L\left[\frac{1}{\sqrt{t}}\right] =$

Problem 2

(13 Marks)

Use Laplace transforms to solve the initial value problem:

$$y'' + 2y = \delta(t - 1) + u\left(t - \frac{\pi}{2}\right) \sin t, \quad y(0) = y'(0) = 1,$$

where $u(t)$ is the unit step function with a step at $t = 0$.

$$y(t) =$$

Problem 3

(12 Marks)

A small mass can move along the x -axis between two large masses located at $x = -1$ and $x = +2$. When subjected to a linear drag its position $s(t)$ obeys the differential equation:

$$s''(t) = \frac{-1}{(s+1)^2} + \frac{4}{(s-2)^2} - s'(t), \quad -1 < s < 2.$$

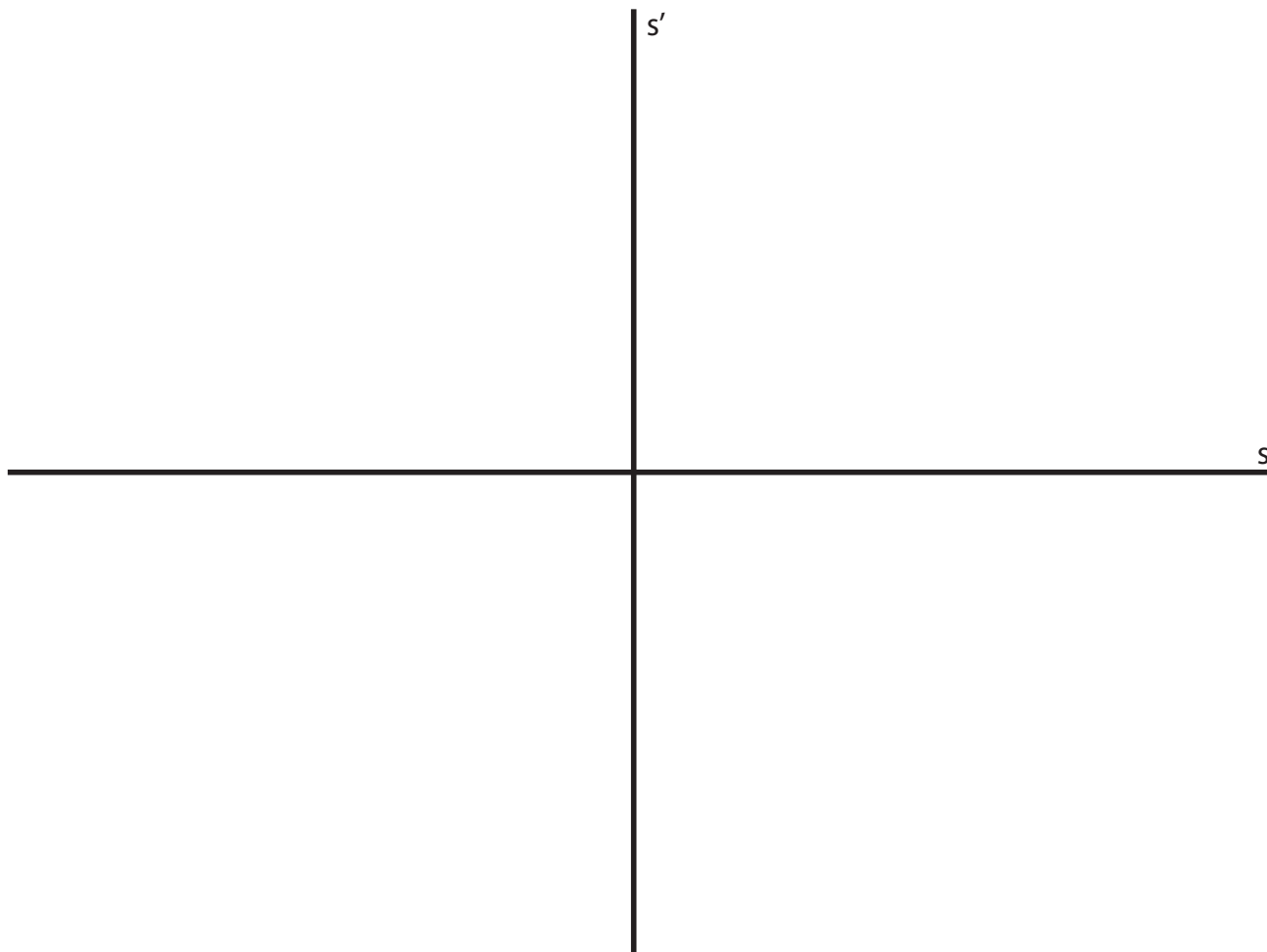
- (a) Convert this equation to a first-order nonlinear system valid for $-1 < s < 2$.

- (b) Find the equilibrium solution of the first order system from (a).

- (c) Classify the equilibrium solution.

The equilibrium point is a

(d) Sketch the phase plane (vector field) and include several possible solution curves.



Problem 4

(12 Marks)

(a) Find all values of m so that $y = t^m$ satisfies $t^2y'' - 6y = 0$.

$m =$

(b) Consider the problem $t^2y'' - 6y = t^3$. Rewrite this as a first order linear system.

(c) Find a fundamental matrix for the system.

- (d) Use variation of parameters to find the general solution of the forced problem in part (b).

$$y(t) =$$

END OF EXAM

Table of Laplace Transforms

$f(t)$	$\mathcal{L}[f(t)] = F(s)$	$f(t)$	$\mathcal{L}[f(t)] = F(s)$
1	$\frac{1}{s}$ (1)	te^{at}	$\frac{1}{(s-a)^2}$ (13)
$e^{at}f(t)$	$F(s-a)$ (2)	$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$ (14)
$\mathcal{U}(t-a)$	$\frac{e^{-as}}{s}$ (3)	$e^{at} \sin kt$	$\frac{k}{(s-a)^2 + k^2}$ (15)
$f(t-a)\mathcal{U}(t-a)$	$e^{-as}F(s)$ (4)	$e^{at} \cos kt$	$\frac{s-a}{(s-a)^2 + k^2}$ (16)
$t^n f(t)$	$(-1)^n \frac{d^n F(s)}{ds^n}$ (5)	$e^{at} \sinh kt$	$\frac{k}{(s-a)^2 - k^2}$ (17)
$f'(t)$	$sF(s) - f(0)$ (6)	$e^{at} \cosh kt$	$\frac{s-a}{(s-a)^2 - k^2}$ (18)
$f^{(n)}(t)$	$s^n F(s) - s^{n-1}f(0) - \dots - f^{(n-1)}(0)$ (7)	$t \sin kt$	$\frac{2ks}{(s^2 + k^2)^2}$ (19)
$\int_0^t f(x)g(t-x)dx$	$F(s)G(s)$ (8)	$t \cos kt$	$\frac{s^2 - k^2}{(s^2 + k^2)^2}$ (20)
t^n ($n = 0, 1, 2, \dots$)	$\frac{n!}{s^{n+1}}$ (9)	$t \sinh kt$	$\frac{2ks}{(s^2 - k^2)^2}$ (21)
$\sin kt$	$\frac{k}{s^2 + k^2}$ (10)	$t \cosh kt$	$\frac{s^2 - k^2}{(s^2 - k^2)^2}$ (22)
$\cos kt$	$\frac{s}{s^2 + k^2}$ (11)		
e^{at}	$\frac{1}{s-a}$ (12)		

Trig identities

$$\begin{aligned} \sin(A \pm B) &= \sin A \cos B \pm \sin B \cos A \\ \cos(A \pm B) &= \cos A \cos B \mp \sin A \sin B \end{aligned}$$