THE UNIVERSITY OF BRITISH COLUMBIA
Sessional Examinations – April 2010

MATHEMATICS 215/255

TIME: 2.5 hours

NO AIDS ARE PERMITTED. Note that this exam has **four** pages. All questions are of equal value. You must answer Questions #1-#4 and **either** Question #5 **or** Question #6. If you answer both Question #5 and #6, only the first one will be graded.

#1

(a) Solve the following initial value problem and give the domain of validity of your solution:

$$\frac{dy}{dx} = \frac{1}{y(1+x)}, \quad y(0) = 1.$$

(b) Suppose the number of sheep in a field $S(t)$ at time t can be modelled by the equation

$$\frac{dS}{dt} = kS(n-S) \qquad (1)$$

where the constant $k > 0$ is the growth rate and the constant $n > 0$ is the field's carrying capacity.

(i) Sketch the solution curves of (1).

(ii) Suppose $S(0) = A \geq 0$. Find $\lim_{t\to\infty} S(t)$.

#2

(a) Find a particular solution of the differential equation with b a constant, $-\infty < b < \infty$:

$$x'' + bx' + x = 2\cos t.$$

(b) Solve the initial value problem

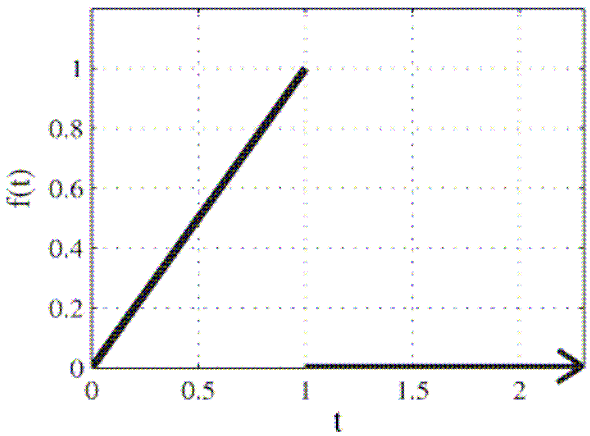
$$x'' + bx' + x = 2\cos t, \quad x(0) = 0, \; x'(0) = 0$$

(i) for $b = 0$;

(ii) for $b = 2$.

#3

(a) Find the Laplace transform of the function $f(t), 0 \le t < \infty,$ sketched below.



(b) For the function $f(t)$ sketched in part (a), solve the following initial value problem:

$$x'' + 2x' + x = f(t), \quad x(0) = -2, x'(0) = 1.$$

(c) Sketch the solution obtained in part (b).

#4.

(a) Consider the system of differential equations given by

$$\begin{aligned} x' &= -x + y, \\ y' &= x + by, \quad -\infty < t < \infty, \end{aligned} \qquad (1)$$

with b a constant.

(i) For certain values of the constant b, one can show that the solutions of (1) are asymptotically stable. Explain what is meant by "asymptotically stable".

(ii) For which values of the constant b are the solutions of (1) asymptotically stable?

(iii) In three separate phase plane diagrams, sketch the solutions of (1) for each of the values $b = -3, 0,$ and $-1.$ In each sketch, indicate the direction of increasing t.

(b) Solve the initial value problem:

$$\begin{aligned} x' &= 2x + 8y + e^t, \\ y' &= -x - 2y + 1, \quad x(0) = y(0) = 0. \end{aligned}$$

ANSWER ONLY ONE OF THE FOLLOWING TWO QUESTIONS.

#5 A pendulum swings according to the nonlinear equation

$$\theta'' + \gamma\theta' + \omega^2 \sin\theta = 0, \qquad (1)$$

where $\theta(t)$ is the angular displacement of the pendulum arm of length ℓ from its equilibrium position at time *t*, the constant $\omega = \sqrt{g/\ell}$ in terms of the gravitational constant *g*, and $\gamma > 0$ is a resistance constant.

(a) Re-write the ODE (1) as a first order nonlinear system in terms of the vector

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \theta \\ \theta' \end{bmatrix}.$$

(b) Find the critical points of the first order nonlinear system. Which points are stable? Justify your answer.

(c) For the stable point closest to (0,0), find the approximating linear system.

(d) Find all values of the constants γ and ω for which this approximating linear system can exhibit

(i) overdamping;

(ii) critical damping;

(iii) underdamping (damped oscillations);

(iv) periodic motion.

For each exhibited case, sketch the solutions of the approximating linear system in the *xy*-phase plane. Show the direction of increasing time.

#6 Consider the nonlinear system

$$\begin{aligned} x' &= y + kx(x^2 + y^2), \\ y' &= -x + ky(x^2 + y^2), \end{aligned} \qquad (1)$$

where k is a constant, $-\infty < k < \infty$.

(a) Find the critical point(s) of (1).

(b) For each critical point, find the approximating linear system.

(c) Sketch the solutions in the *xy*-phase plane for each approximating linear system. Indicate the direction of increasing *t*.

(d) Sketch the solutions for the fully nonlinear system (1) in the *xy*-phase plane. Indicate the direction of increasing *t*. Hint: Consider the use of polar coordinates.

(e) Comment on differences and similarities between the phase plane portraits for the linearizing systems obtained in (c) and the nonlinear systems (1) obtained in (d).

TABLE OF INFORMATION

FUNCTION	**LAPLACE TRANSFORM**
$f(t)$	$F(s)$
$f'(t)$	$sF(s) - f(0)$
1	$\frac{1}{s}$
$u_a(t)f(t-a)$	$e^{-as}F(s)$
$\sin t$	$\frac{1}{s^2+1}$
$\cos t$	$\frac{s}{s^2+1}$
$\int_0^t f(\tau)d\tau$	$\frac{F(s)}{s}$
$tf(t)$	$-F'(s)$
$\int_0^t f(\tau)g(t-\tau)d\tau$	$F(s)G(s)$
$\delta(t-a)$	e^{-as}
$e^{at}f(t)$	$F(s-a)$