

Math 215 Final Exam. April 21st, 2007.

One double-sided sheet of notes is permitted.

No calculators, cellphones, books or other aids are allowed.

This exam has 7 questions worth 100 points in total. Answer all questions.

For each part of each question, draw a box around your final answer.

1. (30 pts)

a. Solve the following system of differential equations with initial conditions.

$$\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix} \mathbf{x} \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

b. $y' = (1+t)e^y$ with $y(0) = 0$. Find $y(t)$.

c. $y'' + 4y = t + \cos 2t$ with $y(0) = 0, y'(0) = 1$. Find $y(t)$.

d. $y'' + 4y = 4\delta(t-6)$ with $y(0) = 0, y'(0) = 1$. Find $y(t)$.

e. Solve the following system of differential equations with initial conditions.

$$\mathbf{x}' = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \mathbf{x} \quad \mathbf{x}(0) = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$$

2. (10 pts) Solve the initial value problem with discontinuous forcing function $g(t)$.

$$y'' + y = g(t) \quad y(0) = 0 \quad y'(0) = 0$$
$$g(t) = \begin{cases} \cos t & 0 \leq t < \pi \\ 0 & \pi \leq t < 2\pi \\ \cos t & t \geq 2\pi \end{cases}$$

Hint: If stuck, think about the simpler problem of $y'' + y = \cos t$ with $y(0) = y'(0) = 0$.

3. (10 pts) Use the method of variation of parameters to find the general solution of the differential equation.

$$y'' - 2y' + y = \frac{e^t}{1+t^2}$$

4. (15 pts) Newton's law of cooling states that the rate of change of temperature of an object is proportional to the difference between its temperature and the surrounding temperature. Assume for this problem that chickens follow this law.

A chicken is initially at room temperature, 25°C . It is then placed in an oven at $t = 0$ and the oven is switched on. The temperature in the oven rises linearly from 25°C at $t = 0$ to 160°C at $t = 30$ minutes.

- Find the temperature $\theta(t)$ of the chicken from $t = 0$ to $t = 30$ minutes. Your answer will contain one unknown parameter, k , the proportionality constant in Newton's law of cooling.
- Make a clear sketch of your solution for large, medium and small values of k .

5. (10pts) The velocity $v(t)$ of a falling object subject to air drag satisfies

$$v' = g - kv^2.$$

- Find the constant-velocity solution $v(t) = V_c$.
- Suppose the object is stationary at $t = 0$. Solve for $v(t)$ for $t > 0$.
- Verify that $\lim_{t \rightarrow \infty} v(t) = V_c$.

6. (15 pts) An electrical circuit is modeled by the following equation:

$$I'' + \frac{1}{2}I' + \frac{1}{C}I = \cos(\omega t), \quad I(0) = 1, \quad I'(0) = 0.$$

$I(t)$ is the current in the circuit and $C > 0$ is the capacitance of one component in the circuit.

- For what values of C does the homogeneous equation exhibit oscillatory behaviour?
- Calculate the particular solution.

7. (10 pts) Let $x(t)$ satisfy the differential equation

$$x'' - x^N = 0, \quad x(0) = 0, \quad x'(0) = 1.$$

- Suppose $N = 0$. Solve the equation and sketch the solution.
- Suppose $N = 1$. Solve the equation and sketch the solution.
- Suppose $N = 2$. Find $\lim_{t \rightarrow \infty} x(t)$, justifying your answer.