The University of British Columbia

Written Final Examination - April 26, 2014

MATH 210 Section 201

Closed book examination — Time: 2.5 hours

Last Name ____________________ **First** ____________________

Signature ____________________ **Student Number** ____________________

Special Instructions:

No memory aids are allowed. No calculators may be used. Show all your work; little or no credit will be given for a numerical answer without the correct accompanying work. If you need more space than the space provided, use the back of the previous page. Where boxes are provided for answers, put your final answers in them.

Rules governing examinations

- Each candidate must be prepared to produce, upon request, a UBCcard for identification.
- Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination.
- Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.
  (a) Having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners.
  (b) Speaking or communicating with other candidates.
  (c) Purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received.
- Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.
- Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

1		18
2		12
3		10
4		12
5		10
6		10
Total		72

[18] **1. True/False Questions**. For each of the questions below, answer whether they are True (T) or False (F) in the answer box provided. Each question is worth 2 marks. You **MUST** provide an explanation to get full credit, and you will not receive **ANY** credit if the reasoning is incorrect.

(a) A negative definite symmetric matrix A has no nontrivial vectors such that $A\mathbf{x} = \mathbf{0}$.

Answer

(b) The following system

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}t} &= v\\ \frac{\mathrm{d}v}{\mathrm{d}t} &= 4u^3v - \sin u + 3\end{aligned}$$

is equivalent to the problem $x'' - 4x^3x' + \sin(x) = 3$ where $'$ indicates differentiation with respect to t.

Answer

(c) I have a box of objects composed of 3 different shapes and each is 1 of 3 colours. Let A be the event of pulling a square and B be the event of pulling a blue object. The probability of pulling a square object or a blue object is

$$P(A \cup B) = P(A) + P(B).$$

Answer

(d) The identity

$$\cos(x) \geq 1 - \frac{x^2}{2}$$

holds on $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Answer

(e) A function $g(x)$ has the following properties

$$g(1) = 3, \qquad g(2) = 3, \qquad g(3) = 7, \qquad g(7) = 2$$

and therefore $x = 2$ is a period-3 orbit of the map $x_{n+1} = g(x_n)$.

Answer

(f) A student claims that they wrote a Newton's method code for solving $f(x) = \cos(x) + 2$ and gets a root converging in 5 iterations. The student's claims are

Answer

(g) The equation

$$\ddot{u} + \sin u = 0$$

has energy $E = \frac{1}{2}\dot{u}^2 + \sin u - u\cos u$.

Answer

(h) A scalar iterative map

$$x_{n+1} = g(x_n)$$

has a fixed point $x = x^*$ such that $g'(x^*) = -4$. This fixed point is unstable.

Answer

(i) The function $f(x) = x + \sec(x)$ satisfies $f(1) > 0$ and $f(2) < 0$. Therefore the bisection method guarantees that a root will be found on $x \in [1, 2]$.

Answer

[12] **2**. In class we talked about how the only value of $\ln(A)$ that we know analytically is $A = 1$ but often we need to know other values.

(a) [4] Use Newton's Method to write the scalar iterative map to solve $x = \ln(A)$ for any $A > 0$. *Hint:* First write the problem as a root finding problem in terms of exponentials.

(b) [4] Define $e_n = x_n - \ln(A)$ to be the error between the Newton iterate and the true value. Write an expression for e_{n+1} that only depends on e_n and constants.

(c) [4] If the error is small, determine the convergence order p for the error, i.e. if $e_{n+1} \approx e_n^3$ then the convergence order is $p = 3$.

[10] **3**. Consider the vector iterative map

$$\begin{aligned} x_{n+1} &= y_n - x_n \\ y_{n+1} &= 2(x_n y_n - \cos(x_n - 1)) \end{aligned}$$

(a) [1] Which of the following two points is a fixed point to this system: $(1, 1)$, $(1, 2)$ or both?

(b) [8] Find a matrix M and numbers a and b such that $J = M^{-1}\mathbb{J}M$ has the form

$$J = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \tag{1}$$

where $\mathbb{J}$ is the Jacobian evaluated at the fixed point(s).

(c) [1]Classify the stability of the fixed point(s)

[12] **4**. Consider two kids playing a video game where their evil wizard characters take turns shooting magic at each other. Player 1 goes first and has a character that can win the game by knocking player 2's character out with a spell that works 2/5 of the time. Player 2 has a character that can win the game by knocking player 1 character's out with a spell that works 4/7 of the time.

(a) [4] Set up the Markov matrix corresponding to this game

(b) [4] What is the probability that someone wins the game after **each** player has had their first turn?

(c) [4] Let the state vector $\mathbf{x}_i$ be the state after turn i of **the game**. Using entries of state vector(s), how would we compute q_3, the probability that **player 2** wins **exactly** on their third turn? *Hint:* Player 2's third turn is not necessarily the third turn of the game.

[10] **5**. Consider the system

$$\frac{\mathrm{d}u}{\mathrm{d}t} = f(u,v), \qquad u(0) = u_0$$
$$\frac{\mathrm{d}v}{\mathrm{d}t} = g(u,v), \qquad v(0) = v_0$$

(a) [4] Find an expression for $\frac{\mathrm{d}^2 u}{\mathrm{d}t^2}$ and $\frac{\mathrm{d}^2 v}{\mathrm{d}t^2}$ involving only f, g, and its partial derivatives.

(b) [6] Develop a second order time-stepping method for approximating the solution $(u(ik),v(ik))$ where k is a predetermined time step and i is an integer. *Note*: The **method** is second order, not the local truncation error.

[10] **6**. Consider an object of mass $m_1 = 3$ at $\mathbf{x}_1 = (1, 0)$ and an object of mass $m_2 = 1$ at $\mathbf{x}_2 = (1, 2)$.

(a) [2] Find the centre of mass of these objects.

(b) [8] After a while, a force field is turned on causing the first object to move. This motion is described by

$$\begin{aligned} \ddot{x}_1 - 3\dot{x}_1 &= 1 - x_1, \qquad & x_1(0) &= 1 \qquad & \dot{x}_1(0) &= 0 \\ \ddot{y}_1 - 3\dot{y}_1 &= 2 - y_1, \qquad & y_1(0) &= 0 \qquad & \dot{y}_1(0) &= 0 \end{aligned}$$

where $x_1(t)$ and $y_1(t)$ are the x and y coordinates of the first object respectively. Write this as a system of first order equations.

Do Not Detach

This page is left intentionally blank and is intended for scrap work. Anything on this page will not be graded

The End - Have a good summer ☺