

The University of British Columbia

Final Examination - December 17, 2015

Mathematics 200

All Sections

Closed book examination

Time: 2.5 hours

Last Name \_\_\_\_\_ First \_\_\_\_\_ Signature \_\_\_\_\_

Student Number \_\_\_\_\_

Special Instructions:

No books, notes, or calculators are allowed. **SHOW YOUR WORK FOR ALL PROBLEMS; NO CREDIT FOR ANSWERS WITHOUT WORK!**

**Student Conduct during Examinations**

- Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
- Candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no candidate shall be permitted to enter the examination room once the examination has begun.
- Candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
- Candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
  - (a) speaking or communicating with other candidates, unless otherwise authorized;
  - (b) purposely exposing written papers to the view of other candidates or imaging devices;
  - (c) purposely viewing the written papers of other candidates;
  - (d) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
  - (e) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)–(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
- Candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
- Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
- Candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

1		9
2		7
3		9
4		8
5		9
6		8
7		9
8		9
Total		68

[9] 1.

- (a) Consider the plane  $4x + 2y - 4z = 3$ . Find all *parallel* planes that are distance 2 from the above plane. Your answers should be in the following form  $4x + 2y - 4z = C$ .

- (b) Find the parametric equation for the line of intersection of the planes

$$x + y + z = 11 \quad \text{and} \quad x - y - z = 13.$$

- (c) Find the tangent plane to

$$\frac{27}{\sqrt{x^2 + y^2 + z^2 + 3}} = 9$$

at the point  $(2, 1, 1)$ .

[7] **2.** A function  $T(x, y, z)$  at  $P = (2, 1, 1)$  is known to have  $T(P) = 5$ ,  $T_x(P) = 1$ ,  $T_y(P) = 2$ , and  $T_z(P) = 3$ .

- (i) A bee starts flying at  $P$  and flies along the unit vector pointing towards the point  $Q = (3, 2, 2)$ . What is the rate of change of  $T(x, y, z)$  in this direction?

- (i) Use linear approximation of  $T$  at the point  $P$  to approximate  $T(1.9, 1, 1.2)$ .

- (iii) Let  $S(x, y, z) = x + z$ . A bee starts flying at  $P$ ; along which unit vector direction should the bee fly so that the rate of change of  $T(x, y, z)$  and of  $S(x, y, z)$  are both zero in this direction?

[9] **3.** Let  $w(s, t) = u(2s + 3t, 3s - 2t)$  for some twice differentiable function  $u = u(x, y)$ .

(a) Find  $w_{ss}$  in term of  $u_{xx}$ ,  $u_{xy}$ , and  $u_{yy}$  (you can assume that  $u_{xy} = u_{yx}$ ).

(b) Suppose  $u_{xx} + u_{yy} = 0$ . For what constant  $A$  will  $w_{ss} = Aw_{tt}$ ?

[8] 4. Find and classify the critical points of  $f(x, y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 4$

[9] **5.** Use Lagrange multipliers to find the minimum and maximum values of  $(x + z)e^y$  subject to  $x^2 + y^2 + z^2 = 6$ .

[8] **6.** Consider the domain  $D$  above the  $x$ -axis and below parabola  $y = 1 - x^2$  in the  $xy$ -plane.

(a) Sketch  $D$ .

(b) Express

$$\iint_D f(x, y) dA$$

as an iterated integral corresponding to the order  $dx dy$ . Then express this integral as an iterated integral corresponding to the order  $dy dx$ .

(c) Compute the integral in the case  $f(x, y) = e^{x-(x^3/3)}$ .

[9] 7. Let  $E$  be the region inside the cylinder  $x^2 + y^2 = 1$ , below the plane  $z = y$  and above the plane  $z = -1$ . Express the integral

$$\int \int \int_E f(x, y, z) dV$$

as three different iterated integrals corresponding to the orders of integration: (a)  $dz dx dy$ , (b)  $dx dy dz$ , and (c)  $dy dz dx$ .

[9] 8. The solid  $E$  is bounded below by the paraboloid  $z = x^2 + y^2$  and above by the cone  $z = \sqrt{x^2 + y^2}$ . Let

$$I = \int \int \int_E z(x^2 + y^2 + z^2) dV$$

- a) Write  $I$  in terms of cylindrical coordinates. Do not evaluate.
- b) Write  $I$  in terms of spherical coordinates. Do not evaluate.
- c) Calculate  $I$ .

**Formula Sheet for Math 200 Final Exam, Fall 2015**

$$\langle a_1, a_2, a_3 \rangle \cdot \langle b_1, b_2, b_3 \rangle = a_1 b_1 + a_2 b_2 + a_3 b_3 = |\mathbf{a}| |\mathbf{b}| \cos \theta$$

$$\text{proj}_{\mathbf{a}} \mathbf{b} = \left( \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2} \right) \mathbf{a} = \left( \frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}} \right) \mathbf{a}$$

$$\text{Det}(\mathbf{a}, \mathbf{b}) = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1 = - \begin{vmatrix} b_1 & b_2 \\ a_1 & a_2 \end{vmatrix} = -\text{Det}(\mathbf{b}, \mathbf{a})$$

$$\text{Det}(\mathbf{a}, \mathbf{b}, \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$\text{Det}(\mathbf{a}, \mathbf{b}, \mathbf{c}) = \mathbf{a} \cdot \left\langle \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}, - \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}, \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \right\rangle = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}).$$

$$\mathbf{b} \times \mathbf{c} = \left\langle \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}, - \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}, \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \right\rangle$$

$$|\mathbf{b} \times \mathbf{c}| = |\mathbf{b}| |\mathbf{c}| |\sin \theta|$$

$$\text{Det}(\mathbf{a}, \mathbf{b}, \mathbf{c}) = -\text{Det}(\mathbf{b}, \mathbf{a}, \mathbf{c}) \quad \text{so} \quad \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = -\mathbf{b} \cdot (\mathbf{a} \times \mathbf{c})$$

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct, \quad \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

$$\mathbf{r}_0 + t\mathbf{v}, \quad (1 - t)\mathbf{r}_0 + t\mathbf{r}_1$$

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0, \quad a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$\text{dist}(ax + by + cz + d_1 = 0, ax + by + cz + d_2 = 0) = \frac{|d_1 - d_2|}{|\langle a, b, c \rangle|}$$

Distance from a point  $P(x_1, y_1, z_1)$  to the plane  $ax + by + cz + d = 0$  is

$$D = |ax_1 + by_1 + cz_1 + d| / \sqrt{a^2 + b^2 + c^2}.$$

$$\begin{aligned}
1/2 &= \sin(30^\circ) = \sin(\pi/6) = \cos(60^\circ) = \cos(\pi/3) \\
1/\sqrt{2} &= \sin(45^\circ) = \sin(\pi/4) = \cos(45^\circ) = \cos(\pi/4) \\
\sqrt{3}/2 &= \cos(30^\circ) = \cos(\pi/6) = \sin(60^\circ) = \sin(\pi/3)
\end{aligned}$$

$$\cos(2\alpha) = 1 - 2\sin^2 \alpha = 2\cos^2 \alpha - 1$$

$$\begin{aligned}
L(x, y) &= f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \\
f(x, y) - f(x_0, y_0) &= z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \\
\Delta f = \Delta z &= f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y \\
df = dz &= f_x(x_0, y_0) dx + f_y(x_0, y_0) dy
\end{aligned}$$

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1} \frac{\partial x_1}{\partial t_i} + \dots + \frac{\partial u}{\partial x_n} \frac{\partial x_n}{\partial t_i}$$

Directional derivative ( $\mathbf{u}$  a unit vector):

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = \langle f_x, f_y, f_z \rangle \cdot \mathbf{u}$$

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

$$f_{xx}, \quad D = f_{xx}f_{yy} - (f_{xy})^2$$

Local min:  $f_{xx} > 0$ ,  $D > 0$ ; local max:  $f_{xx} < 0$ ,  $D > 0$ ; saddle:  $D < 0$ ; degenerate/indeterminate:  $D = 0$ .

Lagrange multipliers to maximize/minimize  $f = f(x, y)$  or  $f = f(x, y, z)$  subject to  $g = C = \text{Constant}$  where  $g = g(x, y)$  or  $g = g(x, y, z)$ :

$$\nabla f = \lambda \nabla g, \quad g = C.$$

Mass and centre of mass, density  $\rho = \rho(x, y)$ :

$$m = \int \int_D \rho(x, y) dA, \quad \bar{x} = (1/m) \int \int_D x\rho(x, y) dA, \quad \bar{y} = (1/m) \int \int_D y\rho(x, y) dA$$

Polar/cylindrical coordinates:  $x = r \cos \theta$ ,  $y = r \sin \theta$ ,

$$dA = dx dy = r dr d\theta, \quad dV = dx dy dz = r dz dr d\theta$$

Spherical (like polar in  $x, y$  with radius  $r = \rho \sin \phi$ ,  $z = \rho \cos \phi$ , so  $\phi$  measures angle with positive  $z$ -axis,  $0 \leq \phi \leq \pi = 180^\circ$ ):

$$x = (\rho \sin \phi) \cos \theta, \quad y = (\rho \sin \phi) \sin \theta, \quad z = \rho \cos \phi, \quad dV = \rho^2 \sin \phi d\rho d\theta d\phi$$