Math 121, Spring Term 2012
Final Exam

April 11^{th},2012

Student number:

LAST name:

First name:

Signature:

Instructions

- Do not turn this page over. You will have 150 minutes for the exam (between 8:30–11:00)
- You may not use books, notes or electronic devices of any kind.
- Except for problem 1, solutions should include full justification and be written clearly, in complete English sentences, showing all your work.
- If you are using a result from the textbook, the lectures or the problem sets, state it properly.
- All parts of a problem have equal value unless noted otherwise.

1		/30
2		/15
3		/16
4		/12
5		/10
6		/12
7		/5
Total		/100

1 Short-form answers (30 points)

Write your answer (simplified, as much as possible) in the box provided. Full marks will be given for the correct answer in the box; show your work for part marks.

a. Evaluate $\int_1^2 \frac{x^2+2}{x^2}$.

Answer:

b. Evaluate $\int_0^{\pi/2} \frac{\cos x}{1+\sin x}\,\mathrm{d}x$.

Answer:

c. Evaluate $\int_{-2012}^{+2012} \frac{\sin x}{\log(3+x^2)}\,\mathrm{d}x$.

Answer:

d. A function $f(x)$ is always positive, has $f(0) = e$ and satisfies $f'(x) = xf(x)$ for all x. Find this function.

Answer:

e. Express $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{ie^{i/n}}{n^2}$ as a definite integral. ***Do not evaluate this integral.***

Answer:

f. Evaluate $\frac{d}{dx}\left[\int_{x^5}^{-x^2} \cos(e^t)\,\mathrm{d}t\right]$

Answer:

g. Evaluate $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n2^n}$**.**

Answer:

h. Evaluate $\sum_{n=1}^{\infty} \frac{n+2}{n!} e^n$**.**

Answer:

i. Evaluate $\lim_{x \to 0} \frac{\sin x - x + \frac{x^3}{6}}{x^5}$**.**

Answer:

j. Find the midpoint rule approximation to $\int_0^\pi \sin x \, \mathrm{d}x$ **with** $n = 3$**.**

Answer:

FULL-SOLUTION PROBLEMS begin here. Give full justification; in particular state the definitions or results from class that you are using.

2 Integrals (15 points)

Evaluate (with justification)

a. $\int_0^3 (x+1)\sqrt{9-x^2}\,\mathrm{d}x$.

b. $\int \frac{4x+8}{(x-2)(x^2+4)}\,\mathrm{d}x$.

Evaluate (with justification)

c. $\int_{-\infty}^{+\infty} \frac{1}{e^x+e^{-x}}\,\mathrm{d}x$.

3 Convergence (16 points)

In each case determine (with justification!) whether the integral or series converges absolutely, converges but not absolutely, or diverges.

a. $\int_{-\infty}^{+\infty} \frac{x}{x^2+1}\,\mathrm{d}x$

b. $\sum_{n=1}^{\infty} \frac{n^2-\sin n}{n^6+n^2}$

In each case determine (with justification!) whether the integral or series converges absolutely, converges but not absolutely, or diverges.

c. $\sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{(n^2+1)(n!)^2}$

d. $\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\log n)^{101}}$

4 Applications (12 points)

a. The probability density function of the random variable X is proportional to

$$f(x) = \begin{cases} \frac{\log x}{x^a} & \textbf{if } x \geq 1 \\ 0 & \textbf{if } x < 1 \end{cases}.$$

where $a > 2$. Find the expectation $\mathbb{E}X$.

b. Let R be the finite region bounded by the lines $y = 0$, $y = x$ and the graph of $y = \cos x$. Let C be the solid obtained by revolving R about the x-axis. Find the volume of C; you may use the constant a such that $a = \cos a$.

5 Convergent Sequence (10 points)

A sequence $\{a_n\}_{n=0}^{\infty} \subset \mathbb{R}$ satisfies the recursion relation $a_{n+1} = \sqrt{3 + \sin a_n}$ for $n \geq 0$.

a. (1 point) Show that the equation $x = \sqrt{3 + \sin x}$ has a solution.

b. (6 points) Show that $\lim_{n\to\infty} a_n = L$, where L is a solution to equation above.

c. (3 points) Show that equation $x = \sqrt{3 + \sin x}$ has a unique solution.

6 A Power series (12 points)

Let $\cosh(x) = \frac{e^x + e^{-x}}{2}$.

a. Find the power series expansion of $\cosh(x)$ **about** $x_0 = 0$ **and determine its interval of convergence. (3 points)**

b. Show that $3\frac{2}{3} \leq \cosh(2) \leq 3\frac{2}{3} + 0.1$**. (5 points)**

c. Show that $\cosh(t) \leq e^{\frac{1}{2}t^2}$ **for all** t**. (4 points)**

7 Last problem (5 points)

Let R be the volume (region of space) obtained by rotating the area between the graph of $f(x) = e^{-|x|}$ and the x-axis about the y-axis. The volume R is filled with a liquid which has density $\sin(\frac{\pi}{2}y)$ at height y above the base plane. Show that the mass M of the liquid satisfies

$$\frac{\pi}{4} \leq M \leq \frac{\pi^2}{8}\,.$$