Math 120 Final

This final is 3 hours, closed book; no notes, calculators, phones, etc. No clarification will be given for any problems; if you believe a problem is ambiguous, interpret it as best you can and write down any assumptions you feel are necessary.

Name: ____________________________

Student #: ____________________________

Problem	Score
1:	/10
2:	/10
3:	/10
4:	/10
5:	/10
6:	/10
7:	/10
8:	/10
9:	/10
Total:	/90

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1. The number $\frac{1+\sqrt{5}}{2}$ is called the golden ratio.

a) What are the two integers closest to the golden ratio? Explain your answer (though you don't have to prove that your answer is correct).

b) Use Newton's method to approximate the golden ratio. Start with one of the two integers you gave above, and iterate Newton's method two times (i.e. you should have your starting guess and then two refinements of this guess). You do not need to simplify fractions.

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(scratch space for problem 1)

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2. a) Prove that if $x > 0$, then

$$\frac{1}{x} = \lim_{h \to 0} \log\left((1 + h/x)^{1/h}\right).$$

b) Prove that

$$\lim_{y \to \infty} \left(1 + \frac{1}{y}\right)^{y} = e.$$

Hint: part (a) might be useful.

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(scratch space for problem 2)

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3. a) Compute the zeroth, first, and second term (plus error term) of the Taylor expansion of $\log(x)$ around the point $c = 1$; i.e. when you apply Taylor's theorem, you should have $n = 3$.

b) Use part (a) to prove that $\log(2) \leq 5/6$.

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(scratch space for problem 3)

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4. Let $y(t)$ be the temperature of a body at time t. Suppose that the body is in a water bath which remains at the constant temperature of 10 degrees. Newton's law of cooling says that the function $y(t)$ obeys the differential equation

$$y'(t) = -k\big(y(t) - 10\big),$$

where k is a constant that depends on how quickly the body conducts heat.

Suppose that the body cools from 200 degrees to 100 degrees in 40 minutes. What is the value of k?

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(scratch space for problem 4)

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5. Let $f(x) = \sin(\log(-x))$ and let $S = \{x \in D(f) \colon f(x) = 0\}$. Does S have a least upper bound? If so, find it (and prove that your answer is correct). If not, prove that S does not have a least upper bound.

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(scratch space for problem 5)

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6. Let $P(x) = a_n x^n + \ldots + a_0$ and $Q(x) = b_m x^m + \ldots + b_0$ be polynomials, and suppose that $a_n > 0,\ b_m > 0$. Prove that

$$\lim_{x\to\infty} \frac{\log(P(x))}{\log(Q(x))}$$

exists (and is a real number). You may use any facts about limits, polynomials, etc. that were proved or stated in class.

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(scratch space for problem 6)

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7. Let $f(x)$ be a function that is differentiable on $[0,1]$. Suppose that $\text{Range}(f)$ is a subset of $[0,1]$ and suppose that there is a number $0 < b < 1$ so that $|f'(x)| < b$ for all $x \in [0,1]$. Prove that there is exactly one number $c \in [0,1]$ with $f(c) = c$.

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(scratch space for problem 7)

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8. Let f be a function whose domain is $\mathbb{R}$. Suppose that for every $x, y \in \mathbb{R}$, $|f(x) - f(y)| \leq |x - y|^2$. Prove that f is a constant function, i.e. there is a number $C \in \mathbb{R}$ so that $f(x) = C$ for all $x \in \mathbb{R}$.

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(scratch space for problem 8)

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9. Let f be a function that is continuous on $\mathbb{R}$. Suppose that for every $x, y \in \mathbb{R}$,

$$f(x + y) = f(x) + f(y).$$

Prove that there is a number $a \in \mathbb{R}$ so that $f(x) = ax$ for all $x \in \mathbb{R}$.

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(scratch space for problem 9)