Be sure that this examination has 12 pages including this cover

The University of British Columbia

Sessional Examinations - December 2012

Mathematics 120

Honours Differential Calculus

Closed book examination — Time: $2\frac{1}{2}$ hours

Name _____ **Signature** _____

Student Number _____ **Instructor's Name** _____

Section Number _____

Special Instructions:

No books, notes, or calculators are allowed.

Rules Governing Formal Examinations

1. Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.

2. Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.

3. No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination.

4. Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.

5. Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:

i. speaking or communicating with other examination candidates, unless otherwise authorized;

ii. purposely exposing written papers to the view of other examination candidates or imaging devices;

iii. purposely viewing the written papers of other examination candidates;

iv. using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,

v. using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s) — (electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).

6. Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.

7. Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

1		16
2		10
3		12
4		12
5		12
6		12
7		12
8		14
Total		100

Marks

[16] **1.** Evaluate the following limits.

(a) $\lim_{x\to\infty} (\ln x)^2 e^{-x}$

(b) $\lim_{x\to 1} \dfrac{\ln(1+x) - x}{x^2}$

(c) $\lim_{x\to\infty} \left[\sqrt{x^2+5x} - \sqrt{x^2-x}\right]$

(d) $\lim_{x\to 0} x \sin^2\left(\frac{1}{x}\right)$

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[10] **2.** The quantities P, Q and R are functions of time and are related by the equation $R = PQ$. Assume that P is increasing instantaneously at the rate of 8% per year and that Q is decreasing instantaneously at the rate of 2% per year. That is, $\frac{P'}{P} = 0.08$ and $\frac{Q'}{Q} = -0.02$. Determine the percentage rate of change for R.

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[12] **3.** Use the formal definition of limit to prove that $\lim_{x \to 0} \cos(3 \sin x) = 1$.

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[12] **4.** The hyperbolic trigonometric functions $\sinh(x)$ and $\cosh(x)$ are defined by

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \cosh(x) = \frac{e^x + e^{-x}}{2}$$

They have many properties that are similar to corresponding properties of $\sin(x)$ and $\cos(x)$. In particular it is easy to see that

$$\tfrac{d}{dx}\sinh(x) = \cosh(x) \qquad \tfrac{d}{dx}\cosh(x) = \sinh(x) \qquad \cosh(x)^2 - \sinh(x)^2 = 1$$

You may use these properties in your solution to this question.

(a) Sketch the graphs of $\sinh(x)$ and $\cosh(x)$.

(b) Define inverse hyperbolic trigonometric functions $\sinh^{-1}(x)$ and $\cosh^{-1}(x)$, carefully specifing their domains of definition. Sketch the graphs of $\sinh^{-1}(x)$ and $\cosh^{-1}(x)$.

(c) Find $\frac{d}{dx}\cosh^{-1}(x)$.

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[12] **5.** An object falls under gravity near the surface of the earth and its motion is impeded by air resistance proportional to its speed. Its velocity v satisfies the differential equation

$$\frac{dv}{dt} = -g - kv$$

where $g,\ k$ are positive constants.

(a) Find the velocity of the object as a function of time t, given that it was v_0 at $t = 0$.

(b) Find $\lim_{t\to\infty} v(t)$.

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[12] **6.** Consider $f(x) = e^{e^x}$.

(a) Find the best linear approximation for f near $x = 0$ (call this $L(x)$).

(b) Find the best quadratic approximation for f near $x = 0$ (call this $Q(x)$).

(c) Prove that $L(x) < Q(x) < f(x)$ for all $x > 0$.

(d) Find an interval of length at most 0.01 that is guaranteed to contain the number $e^{0.1}$.

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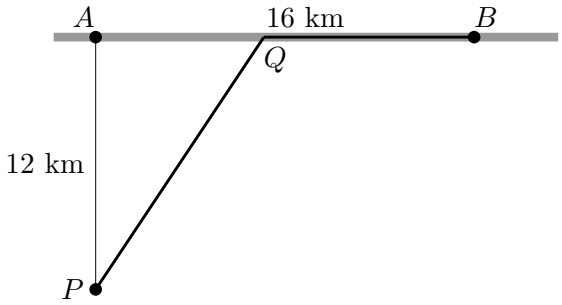
[12] **7.** You are in a dune buggy at a point P in the desert, 12 km due south of the nearest point A on a straight east-west road. You want to get to a town B on the road 16 km east of A. If your dune buggy can travel at an average speed of 15 km/hr through the desert and 30 km/hr along the road, towards what point Q on the road should you head to minimize your travel time from P to B?



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[14] **8.** A function $f(x)$ defined on the whole real number line satisfies the following conditions

$$f(0) = 0 \qquad f(2) = 2 \qquad \lim_{x \to +\infty} f(x) = 0 \qquad f'(x) = K(2x - x^2)e^{-x}$$

for some positive constant K.

(a) Determine the intervals on which f is increasing and decreasing and the location of any local maximum and minimum values of f.

(b) Determine the intervals on which f is concave up or down and the x–coordinates of any inflection points of f.

(c) Determine $\lim_{x \to -\infty} f(x)$.

(d) Sketch the graph of $y = f(x)$, showing any asymptotes and the information determined in parts (a)–(c).

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