The University of British Columbia
Final Examinations - December 2010

MATHEMATICS 120

CLOSED BOOK EXAMINATION **Time:** 3 hours
Notes, calculators are not permitted.
All eight questions are of equal value.

1. Differentiate each of the following functions and in each case give the domain where the derivative exists:

(a) $f(x) = \dfrac{1}{x^2} + \sqrt{x^2 - 1}$;

(b) $g(x) = \pi^x + x^\pi$;

(c) $h(x) = \sin(|x|)$.

2. Evaluate the following limits

(a) $\lim\limits_{x\to 0^+} \dfrac{\ln x}{x}$;

(b) $\lim\limits_{x\to 0} \dfrac{\sin(x^3 + 3x^2)}{\sin^2 x}$;

(c) $\lim\limits_{x\to 0} \dfrac{e^{k\sin(x^2)} - (1 + 2x^2)}{x^4}$, k = const.

3.
(a) Graph the equation $y = f(x) = 4\sin x - 2\cos 2x$, including all important features. In particular, find all local maxima and minima and all inflection points.
(b) Find the maximum and minimum values of $f(x)$ on the interval $[0, \pi]$.

4. Let $f(x) = e^{-x^2}$, $x \geq 0$.
(a) Sketch the graph of the equation $y = f(x)$. Indicate any local extrema and inflection points.
(b) Sketch the graph of the inverse function $y = g(x) = f^{-1}(x)$.
(c) Find the domain and range of $g(x)$.
(d) Evaluate $g'(\tfrac{1}{2})$.

5. Consider an open top rectangular baking pan with base dimensions x centimetres by y centimetres and height z centimetres that is made from A square centimetres of tin plate.. Suppose $y = px$ for some fixed constant p.

(a) Find the dimensions of the baking pan with the maximum capacity (i.e., maximum volume). Prove that your answer yields the baking pan with maximum capacity. Your answer will depend on the value of p.

(b) Find the value of the constant p that yields the baking pan with maximum capacity and give the dimensions of the resulting baking pan. Prove that your answer yields the baking pan with maximum capacity.

6. Consider the problem of approximating $10^{1/3}$ by a rational number.

(a) Use the tangent line to the graph of $y = x^{1/3}$ at $x = 8$ to find an approximate value for $10^{1/3}$. Is the approximation too large or too small? Why?

(b) Use Newton's method applied to the function $f(x) = x^3 - 10$ with the first guess $x_1 = 2$ to find an approximate value for $10^{1/3}$. Is the approximation too large or too small? Why?

7. Consider a curve $y = f(x)$ defined on the interval (a,b). Assume $f''(x)$ exists on (a,b). Let $(x_1, f(x_1))$ be a point on the curve. Find the radius R of the circle with the properties that at $(x_1, f(x_1))$ the circle is both

(a) tangent to $y = f(x)$; and

(b) has the same concavity as $y = f(x)$.

R will depend on $f'(x_1)$ and $f''(x_1)$.

8.

(a) Sketch the graphs of the following functions:

(i) $y = x \ln x$;

(ii) $y = (x-1)\ln x - x$;

(iii) $y = (x+1)\ln x - x$.

(b) Determine the number of local maxima and local minima of the graph of the function

$$y = f(x) = (x+\alpha)\ln x - x$$

in terms of the values of the constant α.