The University of British Columbia

Final Examination - April 18, 2016

Mathematics 105

All Sections

Closed book examination Time: 2.5 hours

Last Name ____________ **First** ________ **Signature** ______________

Student Number ________ **Section Number** ________ **Instructor** ____________

Special Instructions:

No books, notes, or calculators are allowed. A formula sheet is included.

1		42
2		15
3		10
4		14
5		13
6		6
Total		100

[42] **1. Short-Answer Questions**. Put your answer in the box provided but show your work also. Each question is worth 3 marks, but not all questions are of equal difficulty.

(a) Find an equation of the plane that passes through the point $(-1, 0, 3)$ with a normal vector $\langle 3, 5, -2 \rangle$.

Answer:

(b) Find the domain of the function $f(x, y) = \ln(4 - x^2 - y^2)$.

Answer:

(c) Let $f(x, y) = x \sin(xy)$. Find $f_{xx} = \dfrac{\partial^2 f}{\partial x^2}$.

Answer:

(d) Determine whether the following statement is true (you do not need to justify your answer) :

The function $\mathbf{f(x, y)}$ could have both an absolute maximum and an absolute minimum at two different points that are not critical points.

Answer:

(e) Does a left Riemann sum underestimate or overestimate the area of the region under the graph of a function that is positive and decreasing on an interval $[a, b]$? You do not need to justify your answer.

Answer:

(f) Evaluate $\displaystyle\int_{-1}^{5} f(x)\,dx$, where $f(x) = \begin{cases} 2x+4 & \text{if } x \geq 3 \\ 10 & \text{if } x < 3 \end{cases}$.

Answer:

(g) Solve the differential equation: $y'(t) = -0.03y + 2$. You do not need to express your solution as a function of t explicitly.

Answer:

(h) Use substitution rule to find the indefinite integral $\int x^5(x^6 + 23)^{99}\, dx$.

Answer:

(i) Evaluate $\int \frac{\ln x}{x^9}\, dx$.

Answer:

(j) Evaluate $\int \cos^3 x \sin^{-2} x \, dx$.

Answer:

(k) Evaluate $\int \sqrt{16 - x^2} \, dx$. Your answer must be in terms of x, but it is acceptable to leave in inverse trigonometric functions.

Answer:

(l) Let R be the region bounded by the graph of $y = e^{-3x}$ and the x-axis on the interval $[a, \infty)$, where a is a constant.. Find the value of a such that the area of the region R is 1.

Answer:

(m) Evaluate $\displaystyle\sum_{k=0}^{\infty} \left(2 \left(\frac{3}{4} \right)^k + \frac{5}{7^k} \right)$.

Answer:

(n) Find the first three nonzero terms of the Taylor series for the function $f(x) = \dfrac{1}{(1 + x^5)}$ centered at 0.

Answer:

Full-Solution Problems. In questions 2 – 6, justify your answers and show all your work.

[15] **2**.

[3] (a) Compute $\dfrac{d}{dx}\left(\displaystyle\int_x^{x^2} \sin(t^2)\,dt\right)$.

[3] (b) Give the right Riemann sum for $f(x) = x^{\frac{3}{2}}$ on the interval $[2, 6]$ with n subintervals of equal length.

2. [5] (c) Evaluate $\displaystyle\int \frac{81}{x^3 - 9x^2}\, dx$.

[4] (d) Evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \sec^2 x \sqrt{\tan x}\, dx$.

[10] **3**. Use Lagrange multipliers to find the maximum and minimum values of $f(x, y) = y^2 - 4x^2$ subject to the constraint $x^2 + 2y^2 = 4$.

[14] **4.**

(a) Suppose that a continuous random variable X has the probability density function

$$f(x) = \begin{cases} ax^2 & \text{if } 2 \le x \le 3 \\ \dfrac{12}{31} & \text{if } 1 \le x < 2 \end{cases}$$

where a is a constant. Find the value of a.

(b) Let $f(x) = \displaystyle\sum_{k=1}^{\infty} \frac{x^k}{k\,3^k}$, $g(x) = f'(x) = \dfrac{df}{dx}$, and R be the radius of convergence of the power series $f(x)$. Find R and $g(1)$.

[13] **5**. Determine whether the following series converge or diverge.

[2] (a) $\displaystyle\sum_{k=2}^{\infty} \frac{k^2+2k+1}{3k^2+1}$

[3] (b) $\displaystyle\frac{1}{2^2}+\frac{2}{3^2}+\frac{3}{4^2}+\cdots$

5. [3](c) $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k \ln k}$.

[5](d) $\displaystyle\sum_{k=1}^{\infty} \frac{\pi^{k+1}}{k^{\pi} k!}$.

[6] **6**. Evaluate $\displaystyle\int \frac{dx}{x^{2016} - x}$.

MATH 105 Exam Formula Sheet

- **Summation formulas:**

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}, \quad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

- **Trigonometric formulas:**

$$\cos^2 x = \frac{1+\cos(2x)}{2}, \quad \sin^2 x = \frac{1-\cos(2x)}{2}, \quad \sin(2x) = 2\sin x \cos x$$

- **Derivatives of some inverse trigonometric functions:**

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}, \qquad \frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$$

- **Indefinite integrals:**

$$\int \sec x dx = \ln|\sec x + \tan x| + C, \qquad \int \frac{dx}{1+x^2} = \tan^{-1} x + C = \arctan x + C$$

- **Probability:**

  If X is a continuous random variable with probability density function $f(x)$ with $-\infty < x < \infty$, then the expected value $\mathbf{E}(X)$ is given by

$$\mathbf{E}(X) = \int_{-\infty}^{\infty} x f(x) dx$$

- **Some commonly used Taylor series centered at 0:**

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k, \quad \text{for } |x| < 1$$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \quad \text{for } |x| < \infty$$

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}, \quad \cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}, \quad \text{for } |x| < \infty$$

$$\tan^{-1} x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}, \quad \text{for } 1 \geq |x|$$

- **Two important limits:**

$$\lim_{x\to 0} \frac{\sin x}{x} = 1, \qquad \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$$

The End