The University of British Columbia

Final Examination - April 11, 2012

Mathematics 105, 2011W T2

All Sections

Closed book examination Time: 2.5 hours

Last Name ________________ **First** ___________ **SID** ________________

Section number _____ **Instructor name** ________________

Special Instructions:

1. A formula sheet is attached to this exam. No books, notes, or calculators are allowed.
2. Show all your work. A correct answer without accompanying work will get no credit.
3. If you need more space than the space provided, use the back of the previous page. Where boxes are provided for answers, put your final answers in them.

Rules governing examinations

- Each candidate must be prepared to produce, upon request, a UBCcard for identification.
- Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination.
- Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.
  (a) Having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners.
  (b) Speaking or communicating with other candidates.
  (c) Purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received.
- Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.
- Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

Q	Points	Max
1		20
2		20
3		15
4		10
5		10
6		15
7		15
8		45
Total		150

1. (a) Find the radius of convergence of the series

$$\sum_{k=0}^{\infty}(-1)^k 2^{k+1} x^k.$$

(6 points)

(b) You are given the formula for the sum of a geometric series, namely:

$$1 + r + r^2 + \cdots = \frac{1}{1-r}, \qquad |r| < 1.$$

Use this fact to evaluate the series in part (a).

(6 points)

(c) Express the Taylor series of the function

$$f(x) = \ln(1 + 2x)$$

about $x = 0$ in summation notation.

(8 points)

2. This problem contains three numerical series. For each of them, find out whether it converges or diverges. You should provide appropriate justification in order to receive credit.

(a)

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} e^{-\sqrt{k}}$$

(6 points)

(b)

$$\sum_{k=1}^{\infty} \frac{k^4 - 2k^3 + 2}{k^5 + k^2 + k}$$

(6 points)

(c)

$$\sum_{k=1}^{\infty} \frac{2^k (k!)^2}{(2k)!}$$

Recall that $k! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot k$.

(8 points)

3. (a) Compute the following indefinite integral:

$$\int \sin(\ln x)\, dx.$$

(10 points)

(b) Evaluate the following definite integral:

$$\int_0^1 \frac{1}{x^2 - 5x + 6}\, dx.$$

(5 points)

4. Consider the function

$$F(x) = \begin{cases} a & \text{if } x < 0, \\ k \arctan x & \text{if } 0 \le x \le 1, \\ b & \text{if } x \ge 1. \end{cases}$$

(a) Find the values of a, k and b for which F is a valid cumulative distribution function of a continuous random variable. Then sketch the graph of F.

(5 + 2 =7 points)

(b) Let X be a continuous random variable with cumulative distribution function $F(x)$ as given in part (a). Find the probability density function of X.

(3 points)

5. (a) If

$$F(x) = \int_0^x \ln(2 + \sin t)\, dt \quad \text{and} \quad G(y) = \int_y^0 \ln(2 + \sin t)\, dt,$$

find $F'(\frac{\pi}{2})$ and $G'(\frac{\pi}{2})$.

(2+2=4 points)

(b) Now define

$$H(x, y) = \int_y^x \ln(2 + \sin t)\, dt.$$

Find the first partial derivatives of H and use this to compute $\nabla H(\frac{\pi}{2}, \frac{\pi}{2})$.

(2+2+2=6 points)

6. According to market research, the demand curve for a local pizza restaurant satisfies the following relation: if p is the price of a pizza (in dollars), and q is the number of pizzas sold per day, then

$$p^2 + 4q^2 = 800.$$

The restaurant owners want to determine what price the restaurant should charge for each pizza in order to make their daily revenue as high as possible.

(a) Formulate this as a constrained optimization problem, clearly stating the objective function and the constraint.

(5 points)

(b) Use the method of Lagrange multipliers to solve the problem in part (a). There is no need to justify that the solution you obtained is the absolute maximum or minimum. **A solution that does not use the method of Lagrange multipliers will receive no credit, even if the answer is correct.**

(10 points)

7. (a) Find all critical points of the function

$$f(x, y) = xye^y + \frac{1}{2}x^2 - 2.$$

(8 points)

(b) Classify each critical point you found as a local maximum, a local minimum, or a saddle point of $f(x, y)$.

(7 points)

8. Each of the short-answer questions below is worth 5 points. Put your answer in the box provided and show your work. No credit will be given for the answer (even if it is correct) without the accompanying work.

   (a) The Maclaurin series for $\arctan x$ is given by

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1},$$

   which has radius of convergence equal to 1. Use this fact to compute the exact value of the series below:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n}.$$

   (Hint: Note that $3^n = (\sqrt{3})^{2n}$).

Answer:

(b) Find the limit, if it exists, of the sequence $\{a_k\}$, where

$$a_k = \frac{k! \sin^3 k}{(k+1)!}.$$

Answer:

(c) Solve the differential equation

$$\frac{dy}{dx} = xe^{x^2 - \ln(y^2)}.$$

Answer:

(d) Identify and sketch the level curve corresponding to $z = e$ of the function

$$z = e^{x^2+4y^2}.$$

Label the axes of your graph and plot the coordinates of at least four points on the level curve.

Answer: The curve is a (i) parabola (ii) ellipse (iii) circle (iv) hyperbola

(e) There are two boxes containing two balls each. The balls in the first box are numbered -1 and 1, the balls in the second box are numbered 0 and 2. An experimenter draws a ball from each box and observes the number of each ball. Define a random variable X whose value is two times the number of the ball drawn from the first box plus three times the number of the ball drawn from the second box. In other words,

$$X = 2(\text{number observed from box 1}) + 3(\text{number observed from box 2}).$$

Write down all possible values of X and use this to compute the expected value of X.

Answer:
Values of $X =$
$\mathbb{E}(X) =$

(f) Find a bound for the error in approximating

$$\int_0^1 \left[e^{-2x} + 3x^3\right]\, dx$$

using Simpson's rule with $n = 6$ subintervals. There is no need to simplify your answer. **Do not write down the Simpson's rule approximation S_n.**

Answer:

(g) For a certain function $f(x)$, the following equation holds:

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k}{n^2}\sqrt{1 - \frac{k^2}{n^2}} = \int_0^1 f(x)\, dx.$$

Find $f(x)$.

Answer:
$f(x) =$

(h) Evaluate the indefinite integral

$$\int \frac{\sqrt{25x^2 - 4}}{x}\, dx.$$

Answer:

(i) Find the equation of the plane parallel to $3x - y + 4z = 13$ passing through the point $(2, 1, -1)$.

Answer:

The End

Formula Sheet

You may refer to these formulae if necessary.

Summation formulae:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}, \qquad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}, \qquad \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}.$$

Trigonometric formulae:

$$\cos^2 x = \frac{1+\cos(2x)}{2}, \qquad \sin^2 x = \frac{1-\cos(2x)}{2}, \qquad \sin(2x) = 2\sin x \cos x.$$

Simpson's rule:

$$S_n = \frac{\Delta x}{3}\Big(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + ... + 4f(x_{n-1}) + f(x_n)\Big).$$

$$E_s = \frac{K(b-a)(\Delta x)^4}{180}, \quad |f^{(4)}(x)| \le K \text{ on } [a,b].$$

Indefinite Integrals:

$$\int \sec x \, dx = \ln\left|\sec x + \tan x\right| + C.$$

Probability:

- If X is a discrete random variable taking values $x_1, x_2, \cdots, x_m$ with probabilities $p_1, p_2, \cdots, p_m$ respectively, $p_1 + p_2 + \cdots + p_m = 1$, then

$$\mathbb{E}(X) = \sum_{k=1}^{m} x_k p_k, \qquad \mathrm{Var}(X) = \sum_{k=1}^{m} \left[x_k - \mathbb{E}(X)\right]^2 p_k.$$

- If X is a continuous random variable with probability density function $f(x)$, then

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f(x)\, dx, \qquad \mathrm{Var}[X] = \int_{-\infty}^{\infty} (x - \mathbb{E}[X])^2 f(x)\, dx.$$

Approximation using Taylor polynomials:

Let n be a fixed positive integer. Suppose there exists a number M such that $|f^{(n+1)}(c)| \le M$ for all c between a and x inclusive. The remainder in the n^{th}-order Taylor polynomial for f centered at a satisfies

$$|R_n(x)| = |f(x) - p_n(x)| \le M \frac{|x-a|^{n+1}}{(n+1)!}.$$