

The University of British Columbia  
Final Examination - December 16, 2011  
Mathematics 104/184

Time: 2.5 hours

LAST Name \_\_\_\_\_

First Name \_\_\_\_\_ Signature \_\_\_\_\_

Student Number \_\_\_\_\_

MATH 104 or MATH 184 (Circle one) Section Number: \_\_\_\_\_

**Special Instructions:**

No memory aids are allowed. No communication devices allowed. No calculators allowed. Show all your work; little or no credit will be given for a numerical answer without the correct accompanying work. If you need more space than the space provided, use the back of the previous page. Where boxes are provided for answers, put your final answers in them.

**Rules governing examinations**

- Each candidate must be prepared to produce, upon request, a UBCcard for identification.
- Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination.
- Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.
  - (a) Having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners.
  - (b) Speaking or communicating with other candidates.
  - (c) Purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received.
- Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.
- Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

1		42
2		14
3		12
4		12
5		10
6		10
Total		100

[42] **1. Short Problems.** Each question is worth 3 points. Put your answer in the box provided and show your work. No credit will be given for the answer without the correct accompanying work, except for multiple choice questions.

(a) Compute  $\lim_{x \rightarrow 4} \frac{x^2 - 3x - 4}{\sqrt{x} - 2}$ .

Answer:

(b) Find the value of  $a$  for which  $f(x)$  is continuous at  $x = 1$ , where

$$f(x) = \begin{cases} 2x^2 - \ln(x) & \text{if } x \geq 1, \\ x^3 - 4ax & \text{if } x < 1. \end{cases}$$

Answer:

(c) Find  $f'(x)$  where  $f(x) = \frac{(x^2 + 3 \sin^2 x)(e^{x^2})}{x^4 + 7}$ . Do NOT simplify your answer.

Answer:

- (d) Find the equation of the tangent line to  $y = f(x) = \frac{x+3}{2x+1}$  at the point corresponding to  $x = 0$ .

Answer:

- (e) Find the  $x$ -coordinate of the absolute maximum of  $f(x) = x\sqrt{2-x^2}$  on its domain.

Answer:

- (f) Consider a function  $f(x)$  that satisfies  $f'(x) = x[f(x)]^2 + x^2$ . Evaluate  $f''(1)$  if  $f(1) = 2$ .

Answer:

(g) Find the interval(s) on which  $f(x) = x^2 - \ln(x)$  is increasing.

Answer:

(h) You bought a rare stamp collection for 10 million dollars. The auctioneer who sold it to you estimated its value would increase at 12% per year, compounded continuously. If you want to wait until the collection has tripled in value before you sell, how many years will you be waiting?

Answer:

(i) Find the equation of the tangent line to the curve  $xy^2 + 2xy = 8$  at the point  $(1, 2)$ .

Answer:

(j) Let  $f$  and  $g$  be functions such that  $f'(x)$  and  $g'(x)$  are both continuous and increasing for all real numbers  $x$ . Which of the following is ALWAYS true?

- I.  $f + g$  is concave up everywhere.
- II.  $f - g$  is concave up everywhere.
- III.  $f \cdot g$  is concave up everywhere.

- (A) I only.
- (B) II only.
- (C) I and II only.
- (D) I and III only.
- (E) I, II, and III.

Answer:

(k) Let  $g$  be a continuous function on a closed interval  $[0, 1]$ . Let  $g(0) = 1$  and  $g(1) = 0$ . Which of the following is NOT necessarily true?

- (A) There exists a number  $c$  in  $[0, 1]$  such that  $g(c) \geq g(x)$  for all  $x$  in  $[0, 1]$ .
- (B) For all  $a$  and  $b$  in  $[0, 1]$ , if  $a = b$  then  $g(a) = g(b)$ .
- (C) There exists a number  $c$  in  $[0, 1]$  such that  $g(c) = \frac{1}{2}$ .
- (D) There exists a number  $c$  in  $[0, 1]$  such that  $g(c) = \frac{3}{2}$ .
- (E) For all  $c$  in the open interval  $(0, 1)$ ,  $\lim_{x \rightarrow c} g(x) = g(c)$ .

Answer:

(l) If  $f'(a)$  exists, then  $\lim_{x \rightarrow a} f(x)$

- (A) must exist, but there is not enough information to determine it exactly.
- (B) equals  $f(a)$ .
- (C) equals  $f'(a)$ .
- (D) might not exist.
- (E) None of the above.

Answer:

- (m) Find two positive real numbers  $m$  and  $n$  whose product is 50 and whose sum is as small as possible.

Answer:

- (n) Find an approximation to  $(7.5)^{1/3}$  by using the linear approximation to  $f(x) = x^{1/3}$  at  $a = 8$ . Give your answer as a fraction,  $\frac{p}{q}$ , in lowest terms.

Answer:

**Long Problems.** In questions 2 - 6, show your work. No credit will be given for the answer without the correct accompanying work.

[14] **2.** Consider the function

$$f(x) = \frac{x^3 + x^2 - 2x - 3}{x^2 - 3}.$$

Its first and second derivatives are given by

$$f'(x) = \frac{(x^2 - 1)(x^2 - 6)}{(x^2 - 3)^2}, \quad f''(x) = \frac{2x(x^2 + 9)}{(x^2 - 3)^3}.$$

(a) Find all  $x$  such that  $f'(x) = 0$  or  $f'(x)$  does not exist.

(b) Find all  $x$  such that  $f''(x) = 0$  or  $f''(x)$  does not exist.

(c) On which intervals is  $f(x)$  increasing? On which intervals is  $f(x)$  decreasing?

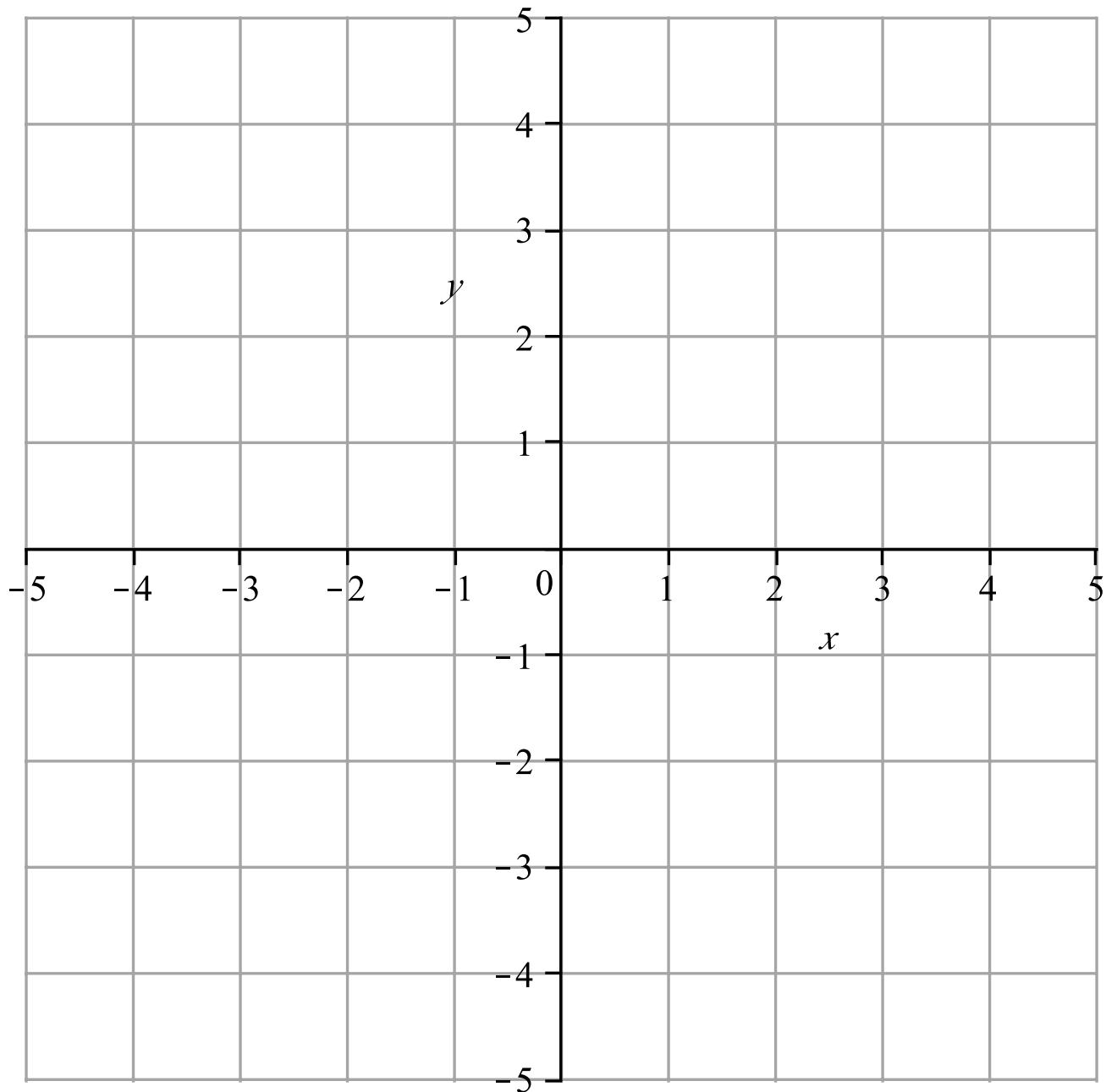
(d) On which intervals is  $f(x)$  concave up? On which intervals is  $f(x)$  concave down?

(e) Find the coordinates of all local maxima, local minima, and inflection points. Be sure to indicate which is which.

(f) Find any asymptotes of the function  $f(x)$  and write their equations.



- (g) Draw the graph of  $f(x)$  on the graph provided. Accurately place all critical points and inflection points, indicate all asymptotes, and make sure your graph correctly shows where  $f(x)$  is increasing and decreasing and correctly shows its concavity.

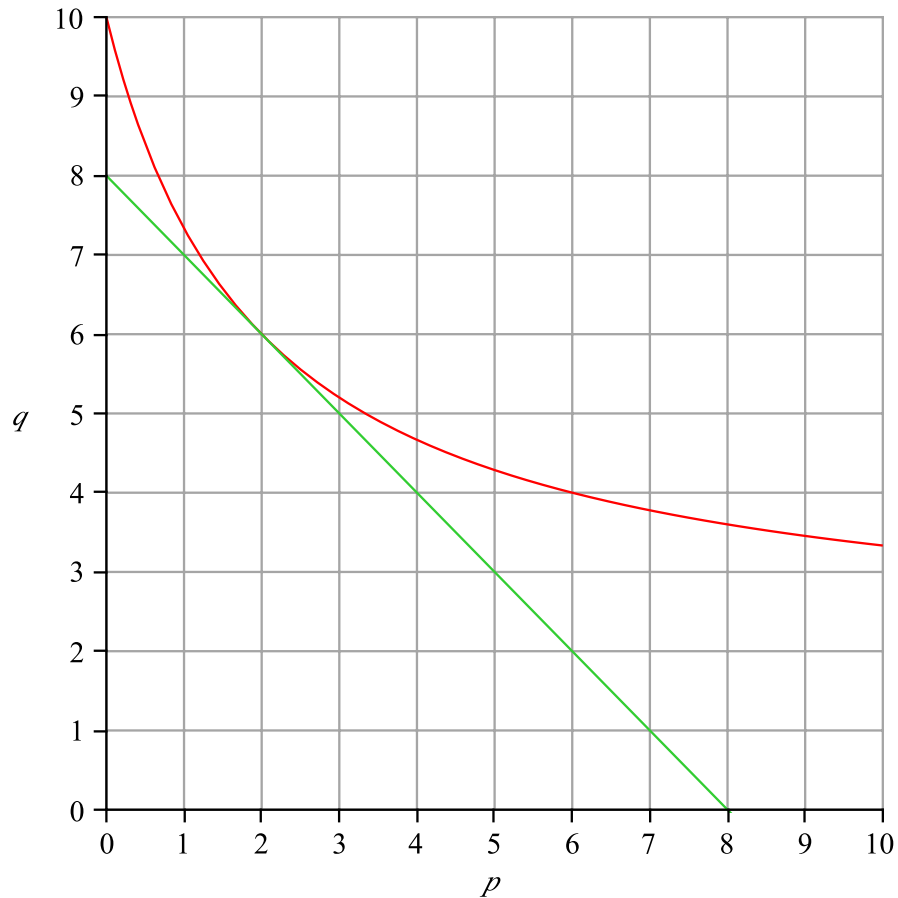


[12] **3.** A box-shaped shipping crate with a square base is designed to have a volume of  $16 \text{ m}^3$ . The material used to make the base costs twice as much per square metre as the material in the sides, and the material used to make the top costs half as much per square metre as the material in the sides. Suppose the material to make the sides costs \$20 per square metre. What are the dimensions of the crate that minimize the cost of the materials?

Answer:

[12] 4. A conical tank of height 5 metres and top radius 4 metres is filled with water and then drains into a cylindrical tank with height 5 metres and radius 4 metres. If the water level in the conical tank drops at a constant rate of 0.5 metres per minute, at what rate does the water level in the cylindrical tank rise when the water level in the conical tank is 3 metres? The volume of a cone of radius  $r$  and height  $h$  is  $V = \frac{1}{3}\pi r^2 h$ . The volume of a cylinder of radius  $R$  and height  $H$  is  $V = \pi R^2 H$ .

[10] 5. In the following figure, the curved graph is a demand curve where  $q$  is the demand and  $p$  is the price in dollars. The straight line is the tangent line to this demand curve at the point  $(p, q) = (2, 6)$ . Recall that the price elasticity of demand is given by  $\epsilon(p) = \frac{p}{q} \frac{dq}{dp}$ .



(a) Compute  $\epsilon(2)$ , the elasticity at price \$2.

Answer:

(b) In order to increase revenue, should the price be raised or lowered from \$2?

Answer:

(c) Use elasticity to estimate the percentage change in price from \$2 required to result in a 5% increase in demand.

Answer:

(d) Describe geometrically (in terms of the graph of the demand curve) how you would find the price that maximizes the revenue. It might be useful to note that the slope of a line connecting a point on this demand curve to the origin is  $q/p$ . Restrict your answer to the box below.

[10] **6.** In this problem, you will approximate  $\ln(0.9)$  in two ways: one using linear approximation and the other using quadratic approximation. You may find it useful to draw the graph of  $\ln(x)$  for  $x$  near 1 to help you answer some parts of this question.

(a) Use the linear approximation to  $f(x) = \ln(x)$  at  $a = 1$  to approximate  $\ln(0.9)$ .

(b) Use the formula  $|\text{Error}| \leq \frac{M}{2}(x - a)^2$ , where  $M > 0$ , to estimate an error bound for your approximation of  $\ln(0.9)$  in part (a). It will be useful to remember how to find such an  $M$ .

(c) Is your approximation for  $\ln(0.9)$  too large or too small? Explain. Use this information together with the error bound you found in part (b) to construct an interval you can guarantee contains the true value of  $\ln(0.9)$ .

- (d) Use the quadratic approximation (second degree Taylor polynomial) to  $f(x) = \ln(x)$  centred at  $a = 1$  to approximate  $\ln(0.9)$ .

- (e) Compare the error bound for the linear approximation to  $\ln(0.9)$  you found in part (b) to the magnitude of the 2nd degree term of the quadratic approximation to  $\ln(0.9)$ . Which do you expect to be larger and why? Restrict your answer to the box below.

**The End**