

This examination has 13 pages of questions excluding this cover

The University of British Columbia

Final Exam - April 21, 2017

Mathematics 103: Integral Calculus with Applications to Life Sciences

201 (Hauert), 202 (Mazumdar), 203 (Hauert), 206 (Kreso), 207 (Lohmann), 208 (Dahlberg),
209 (Dahlberg), 212 (Bade)

Closed book examination

Time: 2.5 hours (150 minutes)

Last Name: _____

First Name: _____

Student Number: _____

Section: circle above

Rules governing examinations:

1. No books, notes, electronic devices or any papers are allowed. To do your scratch work, use the back of the examination booklet. Additional paper is available upon request.
2. You must be prepared to produce your library/AMS card upon request.
3. No student shall be permitted to enter the examination room after 15 minutes or to leave less than 15 minutes before the completion of the examination. Students must ask invigilators for permission to use the washrooms.
4. You are not allowed to communicate with other students during the examination. Students may not purposely view other's written work nor purposefully expose his/her own work to the view of others or any imaging device.
5. At the end of the exam, you will put away all writing implements upon instruction. Students will continue to follow all of the above rules while the papers are being collected.
6. Students must follow all instructions provided by the invigilators.
7. Students are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions
8. Any deviation from these rules will be treated as an academic misconduct. The plea of accident or forgetfulness shall not be received.

I agree to follow the rules outlined above _____
(signature)

Question:	1	2	3	4	5	6	7	8	9	Total
Points:	28	12	14	8	5	9	7	8	9	100
Score:										

Important

1. Simplify all your answers as much as possible and express answers in terms of fractions or constants such as \sqrt{e} or $\ln(4)$ rather than decimals.
2. Show all your work and explain your reasonings clearly!
3. Questions in a section are weighted evenly unless otherwise stated.
4. Formula sheet at the back (you may tear it off and use it for scratch work).
5. Additional sheets of paper for calculations are available upon request.

1. Multiple-choice problems

(Full marks for correct answer. No partial marks.)

- (a) (6 points) Given the following general terms a_n , determine whether the corresponding sequences $\{a_n\}_{n \geq 1}$ are converging, diverging, monotone (increasing or decreasing) and/or bounded. Check *all* boxes that apply. (do not calculate the limit of converging sequences.)

	<i>convergent</i>	<i>divergent</i>	<i>monotone</i>	<i>bounded</i>
<i>i.</i> $\frac{1}{n^2}$:				
<i>ii.</i> $(-1)^n + 1$:				
<i>iii.</i> $-1 + 2^{-n}$:				
<i>iv.</i> $\ln(n)$:				

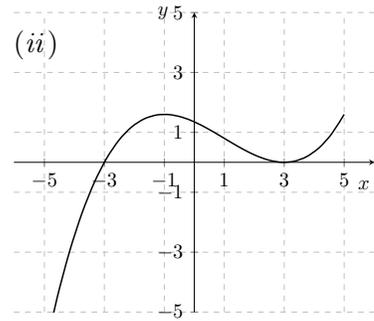
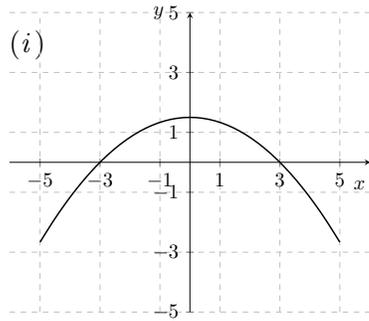
- (b) (4 points) Determine whether the following series converge or diverge. Check appropriate box. (do not calculate the value of converging series.)

	<i>converging</i>	<i>diverging</i>
<i>i.</i> $\sum_{n=1}^{\infty} \frac{1}{n^2}$:		
<i>ii.</i> $\sum_{n=1}^{\infty} \frac{-n^2}{1+n^2}$:		
<i>iii.</i> $\sum_{n=1}^{\infty} \frac{n^2}{n!}$:		
<i>iv.</i> $\sum_{n=0}^{\infty} \frac{1}{\sqrt{3n+4}}$:		

- (c) (4 points) Determine whether the following integrals converge or diverge. Check appropriate box. (do not calculate the integrals.)

	<i>converging</i>	<i>diverging</i>
<i>i.</i> $\int_0^1 \frac{1}{x^3} dx$:		
<i>ii.</i> $\int_e^{\infty} \frac{1}{x(x+1)} dx$:		
<i>iii.</i> $\int_1^{\infty} \frac{\sqrt{\ln(x)}}{x^2} dx$:		
<i>iv.</i> $\int_1^{\infty} \frac{\sqrt{x^3+1}}{x^2} dx$:		

(d) (4 points) For each graph (i) and (ii) identify the graph (A-D), which depicts its antiderivative $A(x) = \int_a^x f(t) dt$ and determine a consistent starting point a of the integral.

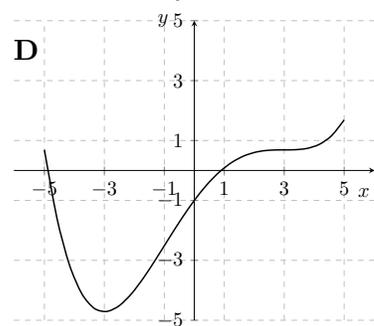
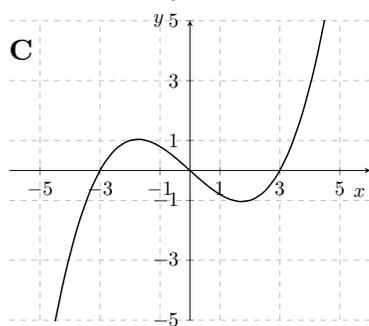
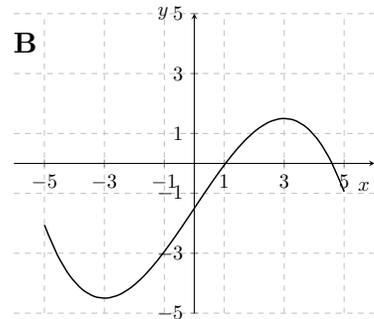
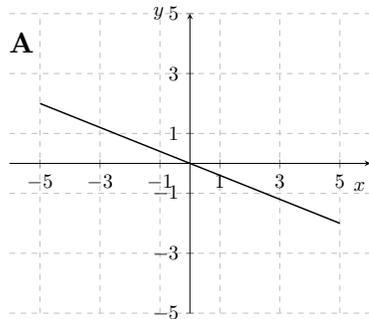


Answer: _____

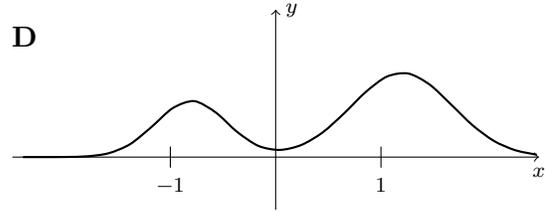
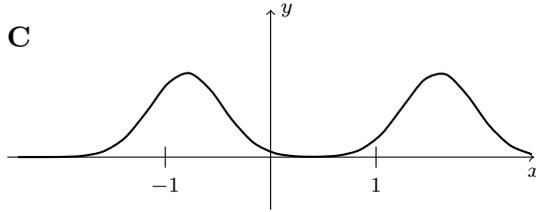
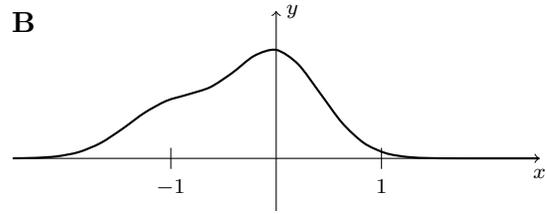
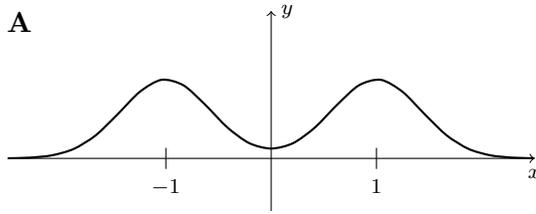
Answer: _____

$a =$ _____

$a =$ _____



(e) (4 points) Plots (A-D) depict probability density functions, *pdf*'s, for $-\infty < x < \infty$ at the same scale. List *all pdf*'s that satisfy the following criteria:
 (\bar{x} denotes the mean and x_{med} the median)



i. $\bar{x} = x_{\text{med}}$? *Answer:* _____

ii. $x_{\text{med}} > \bar{x}$? *Answer:* _____

iii. $\int_{-\infty}^0 p(x) dx > 1/2$? *Answer:* _____

(f) (6 points) Consider the differential equation $\frac{dy}{dt} = 4t\sqrt{y}$. Check *all* solutions in the following list. (Note: different solutions refer to different initial conditions/values.)

- i. $y(t) = t^4$ is a solution
- ii. $y(t) = t^4 + 4t^2 + 4$ is a solution
- iii. $y(t) = t^2$ is a solution

<i>true</i>	<i>false</i>

2. Short-answer-problems

(Full marks for correct answer. Work must be shown for partial marks. Simplify your answers.)

(a) (4 points) Write the following in \sum -notation:

i. $1 + 2 + \frac{9}{4} + \frac{16}{8} + \frac{25}{16} + \frac{36}{32} =$ _____

ii. $1 - 2 + 16 - 2^9 + 2^{16} =$ _____

(b) (4 points) Consider the functions $A_i(x)$ and calculate their derivatives $A'_i(x) = \frac{dA_i(x)}{dx}$.

i. $A_1(x) = \int_0^x e^{-y^2} dy, \quad A'_1(x) =$ _____

ii. $A_2(x) = \int_{e^{-x}}^{e^x} \frac{1}{\ln y} dy, \quad A'_2(x) =$ _____

(c) (4 points) Evaluate $I = \int_0^\pi f(x)f'(x) dx$ using $f(0) = \sqrt{2}, f(\pi) = 2\pi$.

(Work must be shown for full marks.)

ANSWER: $I =$ _____

3. Calculate the following integrals:

(Work must be shown for full marks. Simplify your answers.)

(a) (3 points) $I_1 = \int \frac{\sin(\sqrt{x})}{\sqrt{x}} dx$

ANSWER: $I_1 =$ _____

(b) (3 points) $I_2 = \int y \ln(3y) dy$

ANSWER: $I_2 =$ _____

(c) (4 points) $I_1 = \int_1^2 \frac{1}{x^2} \cos\left(\frac{\pi}{x}\right) dx$

ANSWER: $I_3 =$ _____

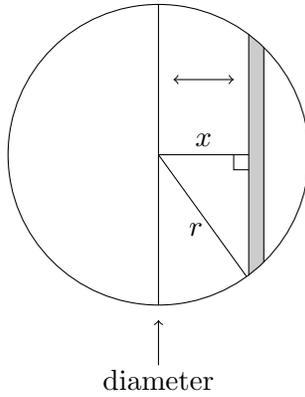
(d) (4 points) $I_2 = \int_0^{\frac{\pi}{4}} \frac{\sin^3 y}{\cos^2 y} dy$

ANSWER: $I_4 =$ _____

4. (8 points) Find *all* x such that the series $\sum_{n=1}^{\infty} \frac{(n-1)^2(x^2-1)^n}{3^n n}$ converges?
(Work must be shown for full marks.)

ANSWER: _____

5. (5 points) In a petri dish with a radius of $r = 3$ cm scientists placed a bacteria killing agent along a diameter. They found the density of bacteria, $\rho(x)$, increased with distance x cm from the diameter according to $\rho(x) = kx$ bacteria/cm² for some constant k . Determine the total population size of bacteria, N , in the petri dish.

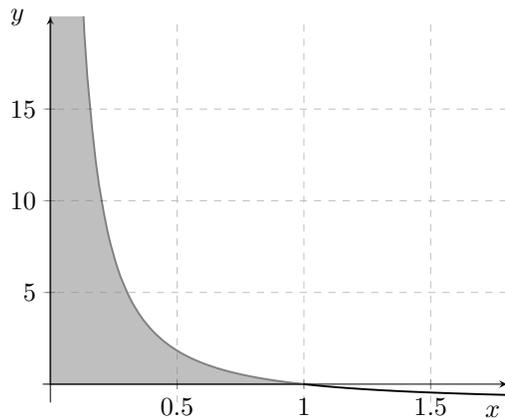


ANSWER: $N =$ _____

6. Consider the function $y = f(x) = x^{-\frac{3}{2}} - 1$.

(Work must be shown for full marks.)

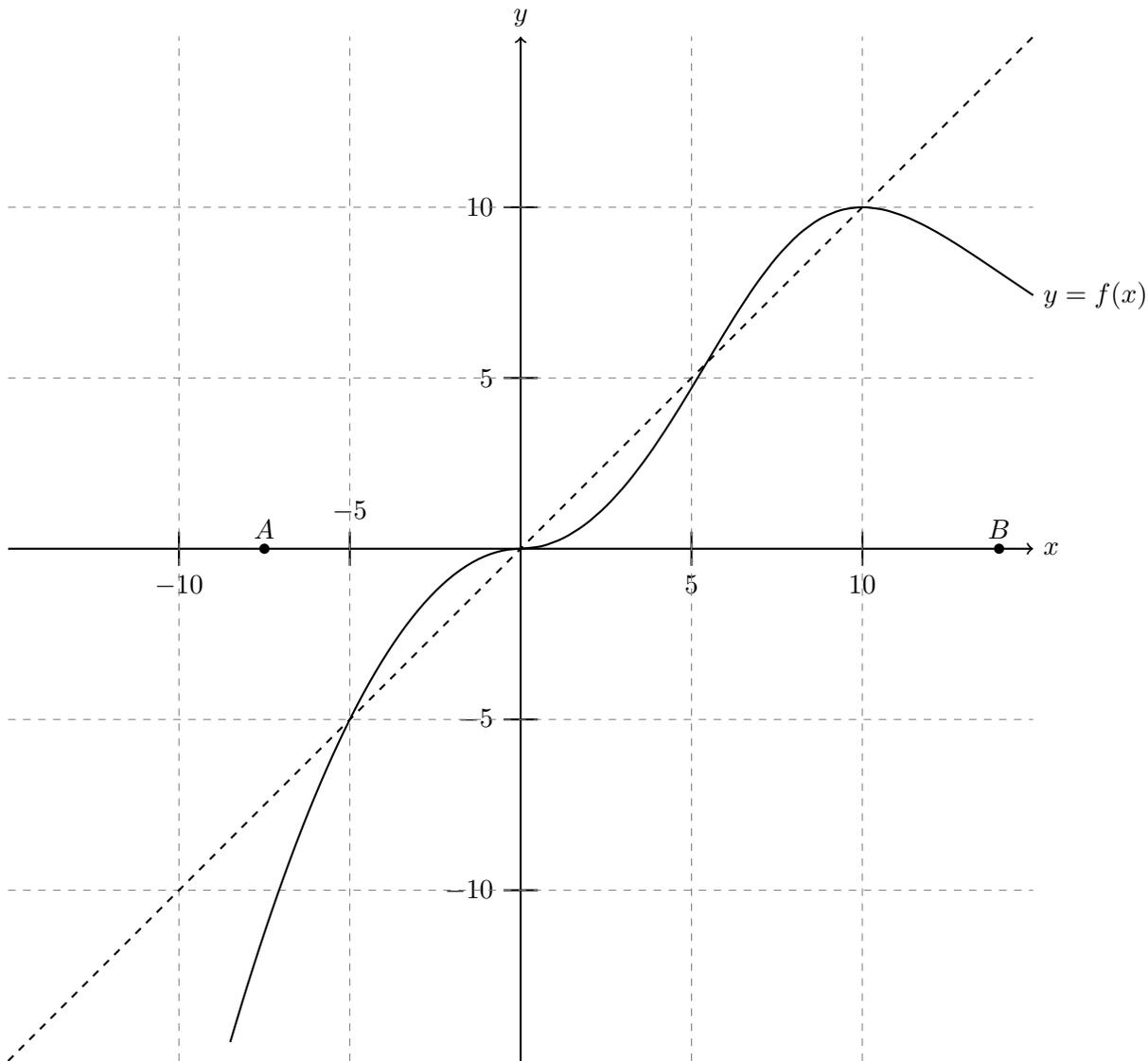
- (a) (4 points) Calculate the area, A , bounded by the x -axis, y -axis and $y = f(x)$ (shaded area in figure) or show that it does not exist.
- (b) (5 points) Calculate the volume of the solid of revolution, V , obtained by rotating the above area (shaded area in figure) around the y -axis or show that it does not exist.



ANSWER: $A =$ _____

$V =$ _____

7. The figure below shows the graph $y = f(x)$ and the diagonal $y = x$ (dashed line).



Use this plot to answer the following questions:

- (a) (2 points) Clearly mark and label *all* fixed points of the iterated map $a_{n+1} = f(a_n)$.
- (b) (3 points) Determine the stability of each fixed point. (Explain your reasoning.)

- (c) (2 points) Use the plot above to draw cobwebs with at least three steps (or until the trajectory exits the graph) for each of the two initial values $a_0 = A$, and B .

8. Suppose the Taylor series $y = \sum_{n=0}^{\infty} a_n x^n$ solves the differential equation $\frac{dy}{dx} + 2y = x^2$ with the initial condition $y(0) = 4$.

(Work must be shown for full marks.)

(a) (6 points) Calculate the first four coefficients a_0 , a_1 , a_2 and a_3 .

ANSWER: $a_0 =$ _____ $a_1 =$ _____ $a_2 =$ _____ $a_3 =$ _____

(b) (2 points) Find the recursive relation $a_n = f(a_{n-1})$ for $n \geq 4$.

ANSWER: $a_n =$ _____

(c) (optional 2 bonus points – save for last) Find the closed formula $a_n = f(n)$ for $n \geq 4$.

ANSWER: $a_n =$ _____

9. The probability density function (*pdf*) for the mortality of a jellyfish (*Turritopsis dohrnii*), $p(x)$, at age x is given by

$$p(x) = \frac{2}{\pi} \frac{1}{1+x^2},$$

for $0 \leq x < \infty$.

- (a) (2 points) Find the probability, $C(x)$, that a jellyfish dies before reaching the age x .

ANSWER: $C(x) =$ _____

- (b) (2 points) Find the median mortality, x_{med} .

ANSWER: $x_{\text{med}} =$ _____

- (c) (3 points) Find the mean mortality, \bar{x} .
(Setup and evaluate the improper integral for the mean mortality.)

ANSWER: $\bar{x} =$ _____

- (d) (1 point) Consider a large colony of *Turritopsis dohrnii* jellyfish. After how many years do we expect that half of the animals of the original colony have died?
(Circle correct answer.)

(i) 1 Year (ii) 2 Years (iii) $\frac{\pi}{2}$ Years (iv) 2π Years (v) Never

- (e) (1 point) The average lifespan of any single jellyfish is arbitrarily long.
(Circle correct answer.)

(i) True

(ii) False

USEFUL FORMULÆ

SUMMATION

$$\sum_{k=1}^N k = \frac{N(N+1)}{2} \quad \sum_{k=1}^N k^2 = \frac{N(N+1)(2N+1)}{6}$$

$$\sum_{k=1}^N k^3 = \left(\frac{N(N+1)}{2}\right)^2 \quad \sum_{k=0}^N r^k = \frac{1-r^{N+1}}{1-r}$$

TRIGONOMETRIC IDENTITIES

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta; \quad \text{for } \alpha = \beta: \quad \sin(2\alpha) = 2 \sin \alpha \cos \alpha$$

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta; \quad \text{for } \alpha = \beta: \quad \cos(2\alpha) = 2 \cos^2 \alpha - 1 = \cos^2 \alpha - \sin^2 \alpha$$

$$\sin^2 \alpha + \cos^2 \alpha = 1 \quad \tan^2 \alpha + 1 = \sec^2 \alpha = \frac{1}{\cos^2 \alpha}$$

SOME USEFUL TRIGONOMETRIC VALUES

$$\sin(0) = 0, \quad \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}, \quad \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}, \quad \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}, \quad \sin\left(\frac{\pi}{2}\right) = 1, \quad \sin(\pi) = 0$$

$$\cos(0) = 1, \quad \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}, \quad \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}, \quad \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}, \quad \cos\left(\frac{\pi}{2}\right) = 0, \quad \cos(\pi) = -1$$

$$\tan(0) = 0, \quad \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}, \quad \tan\left(\frac{\pi}{4}\right) = 1, \quad \tan\left(\frac{\pi}{3}\right) = \sqrt{3}, \quad \tan\left(\frac{\pi}{2}\right) = DNE, \quad \tan(\pi) = 0$$

DERIVATIVES

$$\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}} \quad \frac{d}{dx} \arccos x = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx} \arctan x = \frac{1}{1+x^2} \quad \frac{d}{dx} \tan x = \sec^2 x$$