This examination has 13 pages of questions excluding this cover

The University of British Columbia
Final Exam - April 23, 2015

Mathematics 103: Integral Calculus with Applications to Life Sciences

201 (Doebeli), 202 (Kim), 203 (Hauert), 206 (Sibilla), 207 (Zmurchok), 208 (Chan), 209 (Pinsky)

Closed book examination Time: 2.5 hours (150 minutes)

Last Name: ____________________ **First Name:** ____________________

Student Number: ____________________ **Section:** circle above

Rules governing examinations:

1. No books, notes, electronic devices or any papers are allowed. To do your scratch work, use the back of the examination booklet. Additional paper is available upon request.
2. You must be prepared to produce your library/AMS card upon request.
3. No student shall be permitted to enter the examination room after 15 minutes or to leave less than 15 minutes before the completion of the examination. Students must ask invigilators for permission to use the washrooms.
4. You are not allowed to communicate with other students during the examination. Students may not purposely view other's written work nor purposefully expose his/her own work to the view of others or any imaging device.
5. At the end of the exam, you will put away all writing implements upon instruction. Students will continue to follow all of the above rules while the papers are being collected.
6. Students must follow all instructions provided by the invigilators.
7. Students are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions
8. Any deviation from these rules will be treated as an academic misconduct. The plea of accident or forgetfulness shall not be received.

I agree to follow the rules outlined above ______________________________
(signature)

Question:	1	2	3	4	5	6	7	8	9	10	Total
Points:	19	12	9	6	10	11	9	7	9	8	100
Score:											

Important

1. Simplify all your answers as much as possible and express answers in terms of fractions or constants such as $\sqrt{e}$ or $\ln(4)$ rather than decimals.
2. Show all your work and explain your reasonings clearly!
3. Questions in a section are weighted evenly unless otherwise stated.
4. Formula sheet at the back (you may tear it off and use it for scratch work).
5. Additional sheets of paper for calculations are available upon request.

1. Multiple-choice problems
   (Full marks for correct answer. No partial marks.)

   (a) (3 points) Given the following general terms a_n, determine whether the corresponding sequences $\{a_n\}_{n\geq 1}$ are converging, diverging, and/or bounded. Check *all* boxes that apply. (do not calculate the limit of converging sequences.)

	converging	*diverging*	*bounded*
i. $(-1)^n n$:			
ii. $\dfrac{(-1)^n(n+1)}{n+3}$:			
iii. $\dfrac{n^2+2^n}{e^n+n^e}$:			

   (b) (3 points) Determine whether the following series converge or diverge. Check appropriate box. (do not calculate the value of converging series.)

	converging	*diverging*
i. $\displaystyle\sum_{n=1}^{\infty} \frac{5^n+2^n}{5^n+3^n}$:		
ii. $\displaystyle\sum_{n=1}^{\infty} \frac{(n+1)n^2}{(n+2)^4}$:		
iii. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2 \ln n}$:		

   (c) (3 points) Determine whether the following integrals converge or diverge. Check appropriate box. (do not calculate the integrals.)

	converging	*diverging*
i. $\displaystyle\int_1^{\infty} \frac{x+1}{x^4+x^3+x}\,dx$:		
ii. $\displaystyle\int_2^{\infty} \frac{1}{\ln(x^{10})}\,dx$:		
iii. $\displaystyle\int_0^1 x^{-\frac{4}{5}}\,dx$:		

(d) (4 points) Consider the following differential equations. Check the appropriate box.

	true	*false*
i. $x(t) = -\frac{2}{t^2+2}$ is a solution to $\dfrac{dx}{dt} = tx^2$		
ii. $y(t) = \ln(1 + \frac{t^2}{2})$ is a solution to $\dfrac{dy}{dt} = t\,e^{-y}$		

(e) (6 points) Match Taylor series and functions. Check the appropriate box.

	$\sin x$	$\cos x$	$\frac{1}{(1-x)^2}$	$\ln(1+x)$	e^{-x}
i. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n = 1 - x + \frac{x^2}{2} \dots$					
ii. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)!} x^{2n-1} = x - \frac{x^3}{6} + \frac{x^5}{120} \dots$					
iii. $\displaystyle\sum_{n=0}^{\infty} n x^{n-1} = 1 + 2x + 3x^2 \dots$					

2. Short-answer-problems
(Full marks for correct answer. Work must be shown for partial marks.)

(a) (2 points) Find the telescoping sum $S_N = \sum_{k=1}^{N} \left(\sqrt{k+1} - \sqrt{k}\right)$.

ANSWER: $S_N =$ ____________________

(b) (3 points) Consider the function and its series $f(x) = \dfrac{1}{1-x^3} = a_0 + a_1 x + a_2 x^2 + \ldots$.
Find a_{99} and $f^{(100)}(0)$ (i.e. the 100th derivative of $f(x)$ evaluated at $x = 0$).

ANSWER: $a_{99} =$ ____________________ $f^{(100)}(0) =$ ____________________

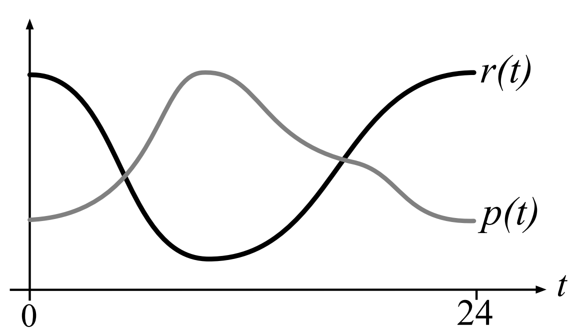
(c) (3 points) Evaluate the following limit: $L = \lim_{x \to 0} \dfrac{\int_0^x \left(e^{t^2} - 1\right) \, dt}{x^3}$.

ANSWER: $L =$ ____________________

(d) (2 points) Express $\int_3^5 f(x)\,dx + \int_4^3 f(x)\,dx + \int_2^4 f(x)\,dx$ as a single integral of the form $\int_a^b f(x)\,dx$.

ANSWER: $a =$ ____________ $b =$ ____________

(e) (2 points) The concentration of a certain hormone in the blood changes *periodically* over 24h. The graph of the production rate $p(t)$ and removal rate $r(t)$ of the hormone are shown below.



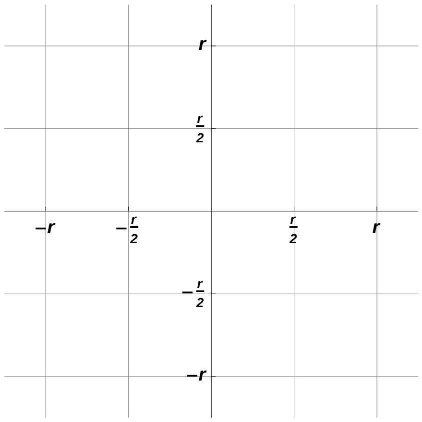
In the figure, clearly mark the times

i. t_1 at which the total concentration of the hormone is highest, and

ii. t_2 at which the hormone concentration is increasing at the fastest rate.

3. The equation of a circle of radius r centred at the origin is $x^2 + y^2 = r^2$.
(Work must be shown for full marks.)

(a) (2 points) The area A of the circle is given by $A = 2\int_{-r}^{r} \sqrt{r^2 - x^2}\, dx$.



Draw a sketch, which clearly identifies the integrand, and shade the area given by $\int_{-r}^{r} \sqrt{r^2 - x^2}\, dx$.

(b) (1 point) Give a reason for the simplification: $2\int_{-r}^{r} \sqrt{r^2 - x^2}\, dx = 4\int_{0}^{r} \sqrt{r^2 - x^2}\, dx$.

(c) (6 points) Evaluate $A = 4\int_{0}^{r} \sqrt{r^2 - x^2}\, dx$. (Hint: use trigonometric substitution.)

ANSWER: $A =$ ____________________

4. (6 points) Consider a swarm of ants distributed over a circular region. At distance r from the centre of the region, the density of the ant population is observed to be $b(r) = \dfrac{1}{r^2+1}$, measured in units of one thousand per square meter. What is the total number of ants within a radius of 2m? (Work must be shown for full marks.)

ANSWER: ______________________

5. Consider the differential equation $\dfrac{dy}{dt} = y^2 - 1.$ (Work must be shown for full marks.)

   (a) (2 points) Find *all* steady states (equilibria).

   (b) (2 points) Determine the stability of *all* steady states (equilibria).
   (Note: formal as well as graphical reasoning is acceptable.)

   (c) (6 points) Solve the differential equation using the initial condition $y(0) = 2$.

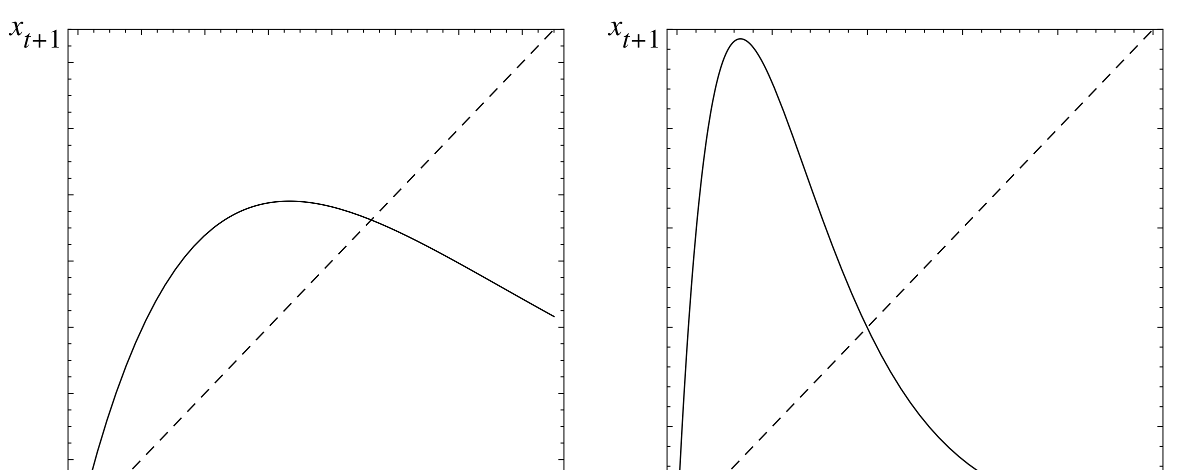
ANSWER: $y(t) =$ ______________________

6. Consider the function $F(x) = \lambda\, x\, e^{-x}$, where $\lambda > 0$ is a parameter. Now consider the iterated map $x_{t+1} = F(x_t)$ for $t = 0, 1, 2, 3, \ldots$. (Work must be shown for full marks.)

   (a) (2 points) Find *all* fixed points (steady states, equilibria).

   (b) (5 points) Determine the range of λ for which each fixed point (steady state, equilibrium) is stable.

   (c) (4 points) Starting at x_0, use cobwebbing to draw x_t up to x_5 in the two graphs below, which show $F(x)$ for two different values of λ.



7. Consider the Normal (or Gaussian) probability density function (*pdf*) given by $f(x) = \sqrt{\dfrac{2}{\pi}}e^{-2x^2}$ for $-\infty < x < \infty$. (Work must be shown for full marks.)

   (a) (2 points) Find the mean $\bar{x}$.

   ANSWER: $\bar{x} =$ ______________________

   (b) (1 point) Find the median $x_{\frac{1}{2}}$.

   ANSWER: $x_{\frac{1}{2}} =$ ______________________

   (c) (2 points) Using the fact that $f(x)$ is a *pdf*, find the integral $I = \displaystyle\int_{-\infty}^{\infty} e^{-2x^2}\, dx$.

   ANSWER: $I =$ ______________________

   (d) (4 points) Find the variance V.
   (Hint: what is the anti-derivative of $x\, e^{-2x^2}$?)

   ANSWER: $V =$ ______________________

8. (7 points) Find *all* x such that the series $\displaystyle\sum_{n=0}^{\infty} \frac{(n-1)^3(x-2)^n}{1+3^n}$ converges?

(Work must be shown for full marks.)

ANSWER: ____________________

9. (9 points) Suppose the power series $y = \sum_{n=0}^{\infty} a_n x^n$ solves the differential equation

$$\frac{dy}{dx} + x\,y = 2$$

with the initial condition $y(0) = -1$. Determine a_0, a_1, a_2 and a_3.
(Work must be shown for full marks.)

10. Integration (Work must be shown for full marks.)

(a) (4 points) Evaluate the integral $I_1 = \displaystyle\int \frac{1+e^x}{1-e^x}\,dx$.

ANSWER: $I_1 =$ ____________________

(b) (4 points) Evaluate the integral $I_2 = \displaystyle\int \cos(\ln x)\,dx$.

ANSWER: $I_2 =$ ____________________

Useful Formulæ

Summation

$$\sum_{k=1}^{N} k = \frac{N(N+1)}{2}$$
$$\sum_{k=1}^{N} k^2 = \frac{N(N+1)(2N+1)}{6}$$
$$\sum_{k=1}^{N} k^3 = \left(\frac{N(N+1)}{2}\right)^2$$
$$\sum_{k=0}^{N} r^k = \frac{1-r^{N+1}}{1-r}$$

Trigonometric identities

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta; \qquad \text{for } \alpha=\beta: \quad \sin(2\alpha) = 2\sin\alpha\cos\alpha$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta; \qquad \text{for } \alpha=\beta: \quad \cos(2\alpha) = 2\cos^2\alpha - 1 = \cos^2\alpha - \sin^2\alpha$$

$$\sin^2\alpha + \cos^2\alpha = 1$$
$$\tan^2\alpha + 1 = \sec^2\alpha = \frac{1}{\cos^2\alpha}$$

Some useful trigonometric values

$$\sin(0) = 0, \quad \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}, \quad \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}, \quad \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}, \quad \sin\left(\frac{\pi}{2}\right) = 1, \quad \sin(\pi) = 0$$
$$\cos(0) = 1, \quad \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}, \quad \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}, \quad \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}, \quad \cos\left(\frac{\pi}{2}\right) = 0, \quad \cos(\pi) = -1$$

Derivatives

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\arccos x = -\frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$$
$$\frac{d}{dx}\tan x = \sec^2 x$$