This examination has 10 pages of questions excluding this cover

The University of British Columbia
Final examination - April 18, 2011

Mathematics 103: Integral Calculus with Applications to Life Sciences

201 (Holmes), 203 (Hauert), 206 (Rolfsen), 207 (Christou), 208 (Lindstrom), 209 (Rolfsen)

Closed book examination Time: 150 minutes (2.5 hours)

Last Name: ____________________ **First Name:** ____________________

Student Number: ________________ **Section:** circle above

Rules governing examinations:

1. No books, notes, electronic devices or any papers are allowed. To do your scratch work, use the back of the examination booklet. Additional paper is available upon request.
2. You should be prepared to produce your library/AMS card upon request.
3. No student shall be permitted to enter the examination room after 30 minutes and to leave within the first 30 minutes or less than 20 minutes before the completion of the examination.
4. You are not allowed to communicate with other students during the examination. Students may not purposely view other's written work nor purposefully expose his/her own work to the view of others or any imaging device.
5. At the end of the exam, you will put away all writing implements and calculators upon instruction. Students will continue to follow all of the above rules while the papers are being collected.
6. Students must follow all instructions provided by the invigilator.
7. Students are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions
8. Any deviation from these rules will be treated as an academic misconduct. The plea of accident or forgetfulness shall not be received.

I agree to follow the rules outlined above ______________________________
(signature)

Question:	1	2	3	4	5	6	7	8	Total
Points:	12	20	10	14	12	10	12	10	100
Score:									

Show all your work and explain your reasonings clearly!

1. (12 points) Short answer problems (only answers are marked)

a. (5) For each of the following series, indicate whether or not they converge:

i. $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n}$ ANSWER: ☐ converges; ☐ does not converge.

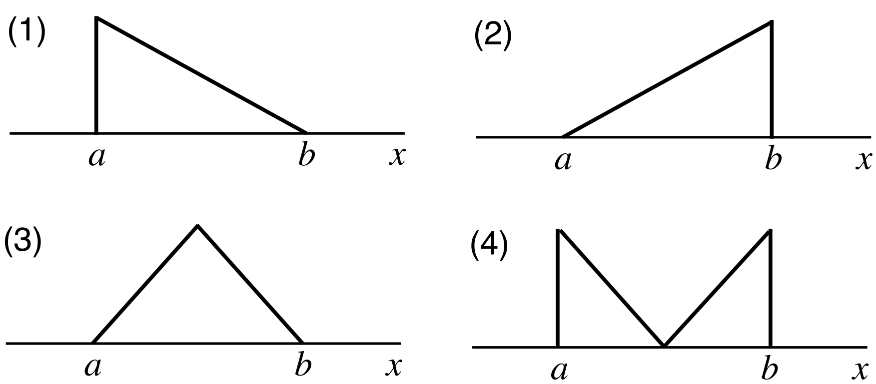
ii. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$ ANSWER: ☐ converges; ☐ does not converge.

iii. $\displaystyle\sum_{n=1}^{\infty} 2^n$ ANSWER: ☐ converges; ☐ does not converge.

iv. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ ANSWER: ☐ converges; ☐ does not converge.

v. $\displaystyle\sum_{n=2}^{\infty} \frac{n}{1-n^3}$ ANSWER: ☐ converges; ☐ does not converge.

b. (5) Consider the following four probability density functions (pdf):



i. Which pdf has the smallest mean? ANSWER: ________

ii. Which pdf has the largest variance? ANSWER: ________

iii. Which pdf has the smallest standard deviation? ANSWER: ________

iv. Which pdf has median larger than the mean? ANSWER: ________

v. Find the maximal probability density in (4).
(Note: the peaks at a and b are of equal heights.) ANSWER: ________

c. (2) A dice is manipulated such that the chance of throwing a 6 is twice as likely as throwing any one of the other numbers (1-5). What is the expected (average) number of throws required to get a 6? Circle the correct answer.

i. 3 **ii.** 3.5 **iii.** 4 **iv.** 6 **v.** 7

2. (20 points) Short problems

a. (4) Evaluate the integral

$$I_1 = \int_{\frac{1}{2}}^{1} \sin(\pi x)\cos(\pi x)dx$$

ANSWER: $I_1 =$ ____________________

b. (6) Evaluate the integral (Hint: use trigonometric identities.)

$$I_2 = \int (\sin x)^3(\cos x)^2 dx$$

ANSWER: $I_2 =$ ____________________

c. (6) Consider the differential equation

$$\frac{dy}{dx} = y - x.$$

Use a Taylor series expansion to find the solution $y(x)$ for the initial value $y(0) = 1$.

ANSWER: $y(x) =$ ____________________

d. (4) A student takes a multiple choice test with 6 questions each of which has 4 possible answers and exactly one is correct. To pass the test at least 5 correct answers are required. (Note: simplify your answers as much as possible but leave fractions and powers.)

i. What is the probability that a student who did not study and randomly checks his/her answers still passes the test?

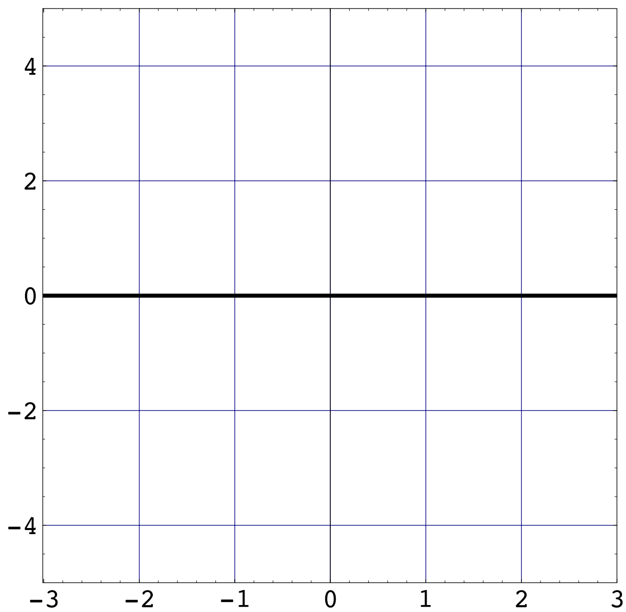
ANSWER: ____________________

ii. With what probability does the student have to get every answer correct in order to get a perfect score with a probability of at least 80%?

ANSWER: ____________________

3. (10 points) Consider the function $f(x) = x(x-2)(x+1)$.

a. (3) Sketch the function.



b. (7) Find the total, finite area A bounded by $f(x)$ and the x-axis.

ANSWER: $A =$ ____________________

4. (14 points) Consider the function $f(x) = e^x$ for $0 \leq x \leq 1$.

   a. (6) Find the volume V_1 of the horn when rotating $f(x)$ around a horizontal axis at $y = 1$.

   ANSWER: $V_1 =$ ____________________

   b. (8) Find the volume V_2 of the bowl when rotating $f(x)$ around the y-axis. (Note: for this problem there are two ways to set the relevant integral up - either one is fine. One way involves rewriting $f(x)$ as $x = g(y)$.)

   ANSWER: $V_2 =$ ____________________

5. (12 points) In a room two different lightbulbs A and B are installed. According to the packaging, lightbulb A fails before time t with probability $F_A(t) = 1 - e^{-t}$ (t in months). Lightbulb B has a probability density function for the failure time given by $p_B(t) = Ce^{-2t}$.

   a. (2) What is the probability density function for the failure time of lightbulb A?

   ANSWER: $p_A(t) =$ ______________________

   b. (2) Determine the constant C of the probability density function for the failure time of lightbulb B?

   ANSWER: $C =$ ______________________

   c. (4) What is the probability that both lightbulbs are still working after t months?

   ANSWER: ______________________

   d. (4) Which light bulb has the longer expected lifetime? (i.e. longer average time to failure.) Show your reasoning.

   ANSWER: ______________________

6. (10 points) Consider the differential equation

$$\frac{dy}{dx} = (y^2 - 1)x.$$

a. (3) Find the steady state solution(s).

ANSWER: ____________

b. (7) Solve for $y(t)$ given the initial value $y(0) = 0$.

ANSWER: $y(t) =$ ____________

7. (12 points) Taylor series

a. (4) Find the first three non-zero terms of the Taylor series for $f(x) = \ln(1+x)$ (around $x = 0$).

ANSWER: ____________________

b. (4) The following definite integral cannot be solved analytically: $\int_0^1 \frac{\sin x}{x} dx$.

Approximate the integral based on the first three non-zero terms of the Taylor series (around $x = 0$).

(Hint: Taylor series for $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$.)

ANSWER: ____________________

c. (4) Find the Taylor series for the function $f(x) = x \cos x$. Write your answer in summation notation. (Hint: the Taylor series for $\sin x$ may be helpful.)

ANSWER: ____________________

8. (10 points) In a lab a bacterial colony is grown in a petri dish. The colony is circular and the area A covered by the colony increases at a rate proportional to its circumference C (because of the high nutrient concentration in the surrounding area):

$$\frac{dA}{dt} = k \cdot C.$$

One day ($t = 0$) the colony was observed to cover an area of 1 $[mm^2]$. The next day the colony had grown to cover 2 $[mm^2]$. What area does the colony cover after T days? (Assume that the petri dish is large enough such that the colony never reaches the boundary.)

ANSWER: $A(T) =$ ____________________

Useful Formulæ

Summation

$$\sum_{k=1}^{N} k = \frac{N(N+1)}{2}, \qquad \sum_{k=1}^{N} k^2 = \frac{N(N+1)(2N+1)}{6}, \qquad \sum_{k=1}^{N} k^3 = \left(\frac{N(N+1)}{2}\right)^2$$

$$\sum_{k=0}^{N} r^k = \frac{1-r^{N+1}}{1-r}, \qquad \sum_{k=0}^{\infty} k r^{k-1} = \frac{1}{(1-r)^2}$$

Trigonometric identities

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta; \qquad \text{for } \alpha=\beta: \quad \sin(2\alpha) = 2\sin\alpha\cos\alpha$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta; \qquad \text{for } \alpha=\beta: \quad \cos(2\alpha) = 2\cos^2\alpha - 1$$

$$\sin^2\alpha + \cos^2\alpha = 1$$

$$\tan^2\alpha + 1 = \sec^2\alpha = \frac{1}{\cos^2\alpha}$$

Some trigonometric values

$$\sin(0) = 0, \quad \sin(\frac{\pi}{6}) = \frac{1}{2}, \quad \sin(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}, \quad \sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}, \quad \sin(\frac{\pi}{2}) = 1, \quad \sin(\pi) = 0$$

$$\cos(0) = 1, \quad \cos(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}, \quad \cos(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}, \quad \cos(\frac{\pi}{3}) = \frac{1}{2}, \quad \cos(\frac{\pi}{2}) = 0, \quad \cos(\pi) = -1$$

Derivatives

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}, \qquad \frac{d}{dx}\arccos x = -\frac{1}{\sqrt{1-x^2}}, \qquad \frac{d}{dx}\arctan x = \frac{1}{1+x^2}$$

Moments of a probability density function

$$M_k = \int_a^b p(x) x^k dx$$