Full Name: ______________________ Signature: ______________________

Student Number: ______________________ Section: ______________________

Math 103 Final Exam April 2006 2.5 hours.

- No calculators, books, notes, or electronic devices of any kind are permitted.
- Unless otherwise indicated, show all your work. Answers not supported by calculations or reasoning may not receive credit. Messy work will not be graded. **Read each question carefully** to be sure you are answering the question being asked.

Rules governing formal examinations:

1. Each candidate must be prepared to produce, upon request, a Library/AMS card for identification;
2. Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions;
3. No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination;
4. Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action;
   (a) Having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners;
   (b) Speaking or communicating with other candidates;
   (c) Purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received;
5. Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator; and
6. Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

Problem #	Value	Grade
1	6	
2	5	
3	12	
4	6	
5	7	
6	7	
7	7	
Total	50	

Formulae

$$\sum_{k=1}^{N} k = \frac{N(N+1)}{2}, \quad \sum_{k=1}^{N} k^2 = \frac{N(N+1)(2N+1)}{6}, \quad \sum_{k=1}^{N} k^3 = \left(\frac{N(N+1)}{2}\right)^2, \quad \sum_{k=0}^{N} r^k = \frac{1-r^{N+1}}{1-r}$$

$$\int_a^b f(x)\ dx = \lim_{\Delta x \to 0} \sum_{k=1}^{N} f(x_k)\Delta x, \quad x_k = a + k\Delta x, \quad \Delta x = (b-a)/N$$

$$\int_a^b F'(x)\ dx = F(b) - F(a), \quad \int_a^b udv = uv\Big|_a^b - \int_a^b vdu$$

$$V_{shell} = \int_a^b 2\pi x f(x)\ dx,\ V_{disk} = \int_a^b \pi f(x)^2\ dx, \quad L = \int_a^b \sqrt{1 + f'(x)^2}\ dx$$

$$\sin^2\theta + \cos^2\theta = 1, \quad \tan^2\theta + 1 = \sec^2\theta, \quad \cos^2\theta = \frac{1+\cos 2\theta}{2}, \quad \sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$\sin(A+B) = \sin A\cos B + \sin B\cos A, \quad \cos(A+B) = \cos A\cos B - \sin A\sin B,$$

$$\int \sec\theta d\theta = \ln|\sec\theta + \tan\theta| + C, \quad \int \sec^3\theta d\theta = \frac{1}{2}\left(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right) + C$$

$$\int \frac{1}{1+u^2} du = \arctan u + C, \quad \int \frac{1}{\sqrt{1-u^2}} du = \arcsin u + C$$

$$\frac{d}{dx}(\tan x) = \sec^2 x, \quad \frac{d}{dx}(\cot x) = -\csc^2 x, \quad \frac{d}{dx}(\sec x) = \tan x\sec x, \quad \frac{d}{dx}(\csc x) = -\cot x\csc x,$$

$$\bar{x} = \mathbf{E}x = \sum_{k=1}^{n} x_k p(x_k), \quad \mathbf{V}x = \sum_{k=1}^{n} x_k^2 p(x_k) - (\mathbf{E}x)^2,$$

$$\bar{x} = \mathbf{E}x = \int_a^b xp(x)\ dx, \quad \mathbf{V}x = \int_a^b x^2 p(x)\ dx - (\mathbf{E}x)^2$$

$$F(x) = \int_a^x p(x)\ dx, \quad F(x_{\text{median}}) = \frac{1}{2}$$

$$p(E_1 \text{ and } E_2) = p(E_1)p(E_2|E_1), \quad p(E_1 \text{ or } E_2) = p(E_1) + p(E_2) - p(E_1 \text{ and } E_2)$$

$$C(n,k) = \frac{n!}{(n-k)!k!}, \quad P(n,k) = \frac{n!}{(n-k)!},$$

$$p(\text{k heads out of n tosses}) = C(n,k)p^k(1-p)^{n-k}$$

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$$

1. For each of the following multiple-choice questions, there is one correct answer. Circle the corresponding (roman) number.

(a) Use the graph of $g(x)$ below to determine the correct set of values of $G(x) = \int_0^x g(t)\, dt$.

i.

x	0	2	4	6	8
$G(x)$	0	2	4	1	0

ii.

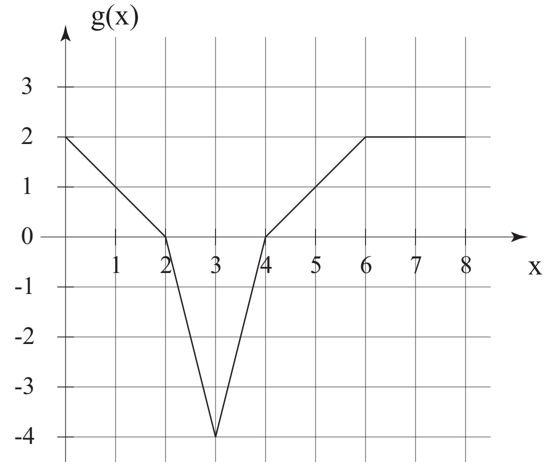
x	0	2	4	6	8
$G(x)$	0	-2	-2	0	4

iii.

x	0	2	4	6	8
$G(x)$	2	0	0	2	2

iv.

x	0	2	4	6	8
$G(x)$	0	2	-2	0	4



(b) A biased coin with $P(H) = 1/10$ is tossed 4 times. What is the probability of getting at least one H?

i. $1 - (9/10)^4$
ii. $(1/10)^4$
iii. $(1/10)(9/10)^3$
iv. $1 - (1/10)^4$
v. $(9/10)^4$

(c) A calcium channel in a cardiac muscle cell opens and closes in a random manner. Before receiving a signal for contraction, the channel is closed. The channel opens t milliseconds after the signal is received where t has a probability density given by $p(t) = \frac{1}{50}e^{-\frac{t}{50}}$ for $0 < t < \infty$. Which of the following gives the expected time at which the channel opens?

i. $\frac{1}{50}\int_0^\infty e^{-\frac{t}{50}}\, dt$
ii. $\frac{1}{50}\int_0^\infty te^{-\frac{t}{50}}\, dt$
iii. $\frac{1}{50}\int_0^\infty t^2e^{-\frac{t}{50}}\, dt$
iv. $\frac{1}{50}\int_0^\infty (t-50)^2e^{-\frac{t}{50}}\, dt$

(d) The concentration of a protein in a long, thin cylindrically shaped bacterium is given by the function $c(x)$ where c is measured in mol/μm along the long axis of the cell and $0 < x < 3$ is measured in μm. Suppose that when the cell divides, it splits at $x = 3/2\mu$m and that all protein to the left of this point ends up in the left daughter cell and any to the right ends up in the right daughter cell. Determine which of the following statements is necessarily true:

i. If the center of mass of the protein density is to the left of the median of the density, than the cell on the right gets more than half of the total protein.

ii. If the center of mass of the protein density is at $x = 1\mu$m, then the cell on the left gets more than half of the total protein in the mother cell.

iii. If the median of the protein density is at $x = 1\mu$m, then the cell on the left gets more than half of the total protein in the mother cell.

iv. If the mean and the median are at the same point, both cells get the same amount of protein.

(e) $1 - 2x^2$ is the second order Taylor polynomial (around $x = 0$) for which one of the following functions?

i. $1 - \sin(4x)$

ii. e^{-x^2}

iii. $\cos(2x)$

iv. e^{-4x}

v. $\cos(x)$

(f) Which of the following is essentially a form of the Fundamental Theorem of Calculus?

i. The area under the graph of $f(x)$ between $x = a$ and $x = b$ is given by $\int_a^b f(x)\,dx$.

ii. $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x$.

iii. The derivative with respect to time of the position of a molecule is v. The net displacement of the molecule from $t = 0$ to $t = T$ is given by $\int_0^T v(t)\,dt$.

iv. $\int_a^b f(x)\,dx = f'(b) - f'(a)$.

2. For each of the following short questions, write your answer in the box – it will be graded simply right or wrong.

   (a) Calculate $\sum_{k=1}^{10}(k+3)^2$.

   (b) If $\int_1^4 f(x)dx = 2$ and $\int_3^4 f(x)dx = 5$, find $\int_1^3 f(x)dx$.

   (c) If the continuous random variable x has probability density $p(x) = \frac{\pi}{2}\sin(\pi x)$, $0 \leq x \leq 1$, find the mean value of x.

   (d) A bag contains 5 red balls, 3 green balls, and 2 yellow balls. If balls are always replaced into the bag after being drawn, what is the probability of drawing the same colour out of the bag on two successive attempts?

   (e) The Taylor series for the function $\frac{1}{1+x^2}$ around $x = 0$ is

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots.$$

   Use this fact to find the Taylor series for $arctan(x)$ around $x = 0$ (hint: the formula sheet might be useful).

3. Evaluate the following integrals.

(a) $\int \frac{cos(x)}{sin(x)} dx$

(b) $\int \frac{1}{(x-5)(x+1)}\ dx$

(c) $\int_0^2 \sqrt{4-x^2} dx$

(d) $\int_1^2 \ln(x) x dx$

4. You are driving your car at 30 m/sec (approx. 108 km/hr) to catch your flight to Costa Rica for summer holidays. A pedestrian runs across the road, forcing you to brake hard. Suppose it takes you 1 sec to react to the danger, and that when you apply your brakes, you slow down at the rate $a = -10$ m/sec^2. After applying the brakes, how long will it take you to stop? How far will your car move from the instant that the danger is sighted until coming to a complete stop?

5. Consider the region below the graph $y = 1 - x^3$ (and above the x-axis) between $x = 0$ and $x = 1$.

   (a) Find the area of this region.

   (b) Find the volume of the solid "dome" obtained by rotating this region about the y-axis.

   (c) Suppose the density of the solid from part (b) is $p(y) = ky$. Find its mass.

6. Let t be the time (in hours) it takes for a cell division to occur. Suppose t is a continuous random variable on the interval $0 \le t \le 1$ (i.e. the cells always divide in less than 1 hour), with probability density

$$p(t) = C(\sqrt{t} - t), \qquad 0 \le t \le 1.$$

   (a) Find C.

   (b) Find the probability that cell division occurs in less that 30 minutes.

   (c) Find the mean division time.

7. In an experiment involving a bacteria population, $N(t)$ denotes the size of the population (measured in thousands of individuals) as a function of time, starting at $t = 0$. The initial population is $N(0) = N_0$.

   (a) Suppose the population growth is governed by the differential equation

   $$\frac{dN}{dt} = \frac{N}{t+1}.$$

   Find $N(t)$ if $N(0) = 2$.

   (b) Suppose the population growth is governed by the differential equation

   $$\frac{dN}{dt} = \frac{N^2}{(t+1)^2}.$$

   Find $N(t)$ if $N(0) = 2$.

   (c) What happens to the solution from part (b) as $t \to 1$? Can you find an initial population N_0 for which this problem doesn't occur for any time t?