Be sure this exam has 12 pages including this cover

The University of British Columbia

Sessional Examinations – December 2010

Mathematics 102 — Differential Calculus with applications to Life Sciences

Closed book examination Time: $2\frac{1}{2}$ hours

Name: ______________________ **Signature:** ______________________

Student Number: ______________________ **Section:** ______________________

Allowed calculators: Texas Instruments TI 30X (including solar and MultiView versions), Casio FX-260, Sharp EL-510RB. No notes or other aids. Except in Question 1, you must show your work to obtain full credit. Express answers in terms of fractions or constants such as $\sqrt{3}$ or $\ln(4)$ rather than decimals, unless a numerical value is asked for. Where a box is provided, write the answer (but not your work) in the box. The last page contains some helpful formulae.

Problem	total possible	score
1.	18	
2.	11	
3.	8	
4.	10	
5.	6	
6.	13	
7.	10	
8.	5	
9.	9	
10.	10	
total	100	

Section 101: N. Tania
Section 102: M. Willoughby
Sections 103 and 104: R. Israel
Section 105: D. Steinberg
Section 108: C. Hauert
Section 109: I. Rozada

1. For this short-answer question, only the answers (placed in the boxes) will be marked.

(3 points) (a) For the curve $y^2 = 5x^4 - x^2$, find $\dfrac{dy}{dx}$

(3 points) (b) On what interval(s) is the graph of $y = 2x^4 - 4x^3 - 9x^2 - x + 3$ concave down?

(3 points) (c) Find the derivative of $f(x) = \big(\ln(x^2+1)\big)^3$.

(3 points) (d) A certain bacteria culture starts with 1000 bacteria, and the number of bacteria doubles every 10 minutes. If this continues indefinitely, how long (in **hours**) will it take for the population to reach 8×10^{15}?

(3 points) (e) If Newton's Method is used to solve the equation $x^3 + x - 5 = 0$, starting at $x_0 = 1$, what is x_1?

(3 points) (f) A certain function f satisfies $f(3) = 5$ and $f'(3) = 7$. Given these facts, find an approximate numerical value for $f(3.02)$.

2. Suppose the tangent line to the graph of $y = f(x)$ at $x = 1$ is $y = 2x + 1$, and $f(3) = 2$.

(2 points) (a) What are the values of $f(1)$ and $f'(1)$?

$f(1) =$ $\boxed{}$ $f'(1) =$ $\boxed{}$

(3 points) (b) What is the average rate of change of f on the interval $1 \le x \le 3$?

Average rate of change = $\boxed{}$

(3 points) (c) Let $g(x) = 3f(4x + 5)$. Given the information above, there is one number b for which both $g(b)$ and $g'(b)$ can be found. What are b, $g(b)$ and $g'(b)$?

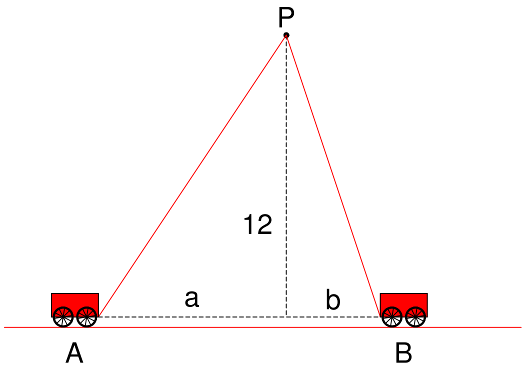
$b =$ $\boxed{}$, $g(b) =$ $\boxed{}$, $g'(b) =$ $\boxed{}$

(3 points) (d) Suppose f has an inverse function f^{-1}. Given the information above, there is one number c for which both $f^{-1}(c)$ and $(f^{-1})'(c)$ can be found. What are c, $f^{-1}(c)$ and $(f^{-1})'(c)$?

$c =$ $\boxed{}$, $f^{-1}(c) =$ $\boxed{}$, $(f^{-1})'(c) =$ $\boxed{}$

(8 points) 3. Use the definition of derivative (no differentiation rules) to find $f'(2)$, where $f(x) = 1/x$.

(10 points) 4. Two carts A and B are connected by a rope 39 metres long that passes over a small stationary pulley P, 12 metres above the height at which the rope is attached to the carts, as shown in the figure. The rope is assumed to form straight lines from the carts to the pulley, with constant total length 39 metres. Cart B is pulled to the right at 2 metres per second. How fast is cart A moving when the point where the rope is attached to cart B is 5 metres to the right of the pulley? *Hint:* If the sum of two things is constant, how are their rates of change related?



$$\frac{da}{dt} = \boxed{}$$

(6 points) 5. When a person breathes, the volume of air in the lungs may be modelled by a function of the form $V(t) = C + A\sin(\omega t + \phi)$, where V is the volume in millilitres and t is time in seconds. Suppose the minimum and maximum volumes are 1400 and 3400 ml respectively, and the maximum rate of change of V is 1200 ml/sec. What is the period of $V(t)$?

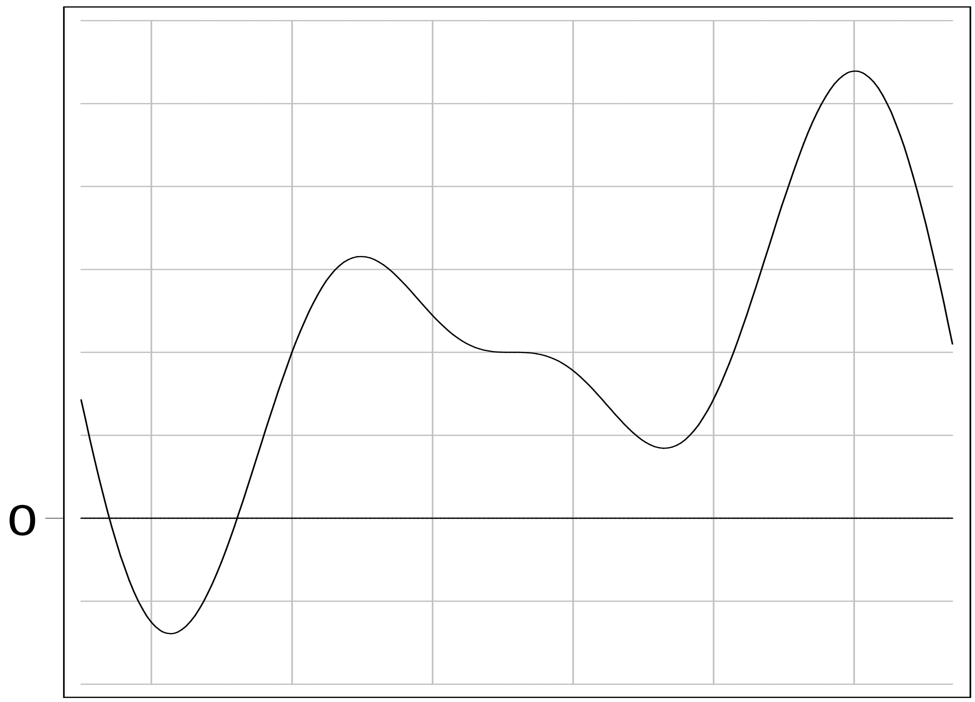
Period= $\boxed{}$

6. On the graph below, mark

(2 points) (a) all critical points (and classify them as local maximum, local minimum or neither)

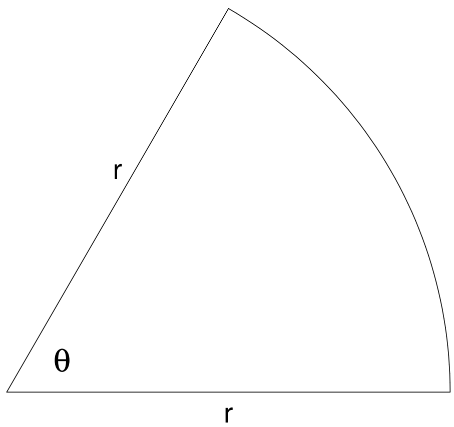
(2 points) (b) all inflection points

(2 points) (c) all intervals on which y is concave up.



(7 points) (d) Sketch the graph of dy/dx on the axes below.

(10 points) 7. A sector of a circle with radius r and opening angle θ has area $A = r^2\theta/2$. Find r and θ for the sector with smallest perimeter, given that the area $A = 9$. Note that the perimeter consists of two radii and a circular arc. Be sure to verify that your answer is the global minimum.



$r =$ $\boxed{}$, $\theta =$ $\boxed{}$

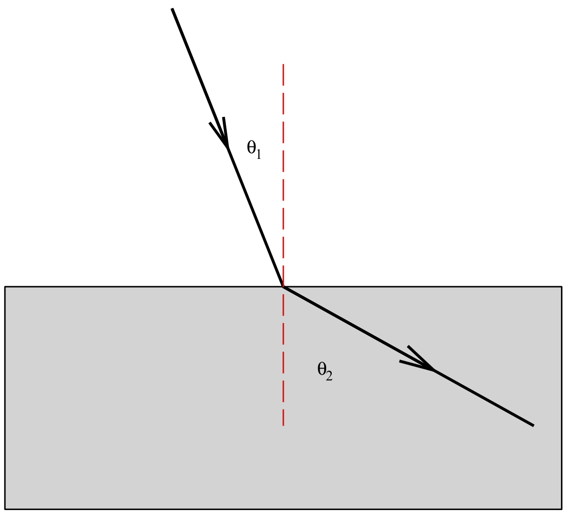
(5 points) 8. After a certain drug is injected into a person's bloodstream, its concentration in the blood decreases at a rate proportional to the existing concentration. If time is measured in hours, the constant k in the differential equation for the concentration is 0.25. If the initial concentration was 0.8 mg of drug per ml of blood, how long does it take for the concentration to decrease to 0.16 mg per ml?

It takes $\boxed{}$ hours

9. When light travels from one medium to another, it is refracted (i.e. changes direction). The relation between the directions of the incoming and outgoing rays is given by Snell's Law:

$$n_2 \sin(\theta_2) = n_1 \sin(\theta_1)$$

Here n_1 and n_2 are constants called the refractive indices of the two media; θ_1 is the angle between the incoming ray and a line perpendicular to the boundary between the media, and θ_2 is the angle between the outgoing ray and that perpendicular line, as shown below. Suppose $n_1 = \sqrt{2}$ and $n_2 = 1$.



(6 points) (a) Suppose we increase θ_1 at a rate of 1/10 radian per second. Find the rate of change of θ_2 at the moment when $\theta_1 = \pi/6$.

$\dfrac{d\theta_1}{dt} =$ $\boxed{}$

(3 points) (b) If θ_1 is too large, Snell's law can't be satisfied, and the light can't enter the second medium. At what value of θ_1 does this start to occur in this example?

$\theta_1 =$ $\boxed{}$

10. Consider the differential equation

$$\frac{dx}{dt} = \frac{9x - x^3}{1 + x^2}$$

(3 points) (a) Find all stable and unstable steady states (equilibria).

Stable: $\boxed{}$ Unstable: $\boxed{}$

(4 points) (b) What value (if any) does $x(t)$ approach as $t \to \infty$ for each of these initial values?

$x(0) = 1$: $\boxed{}$, $x(0) = 4$: $\boxed{}$

(3 points) (c) Use two steps of Euler's method to find a numerical approximation to $x(0.5)$ for the solution with $x(0) = 1$.

$x(0.5) \approx$ $\boxed{}$

Useful Formulae

Length of an arc of a circle:

$$s = r\theta$$

Law of cosines:

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

Trig identities:

$$\begin{aligned}
&\sin^2\theta + \cos^2\theta = 1\\
&\sin(A+B) = \sin A\cos B + \cos A\sin B\\
&\cos(A+B) = \cos A\cos B - \sin A\sin B\\
&\tan\theta = \frac{\sin\theta}{\cos\theta}
\end{aligned}$$

Values:

θ	$\sin\theta$	$\cos\theta$
0	0	1
$\pi/6$	$1/2$	$\sqrt{3}/2$
$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$
$\pi/3$	$\sqrt{3}/2$	$1/2$
$\pi/2$	1	0