Be sure this exam has 12 pages including this cover

The University of British Columbia

Sessional Examinations – December 2009

Mathematics 102 — Differential Calculus with applications to Life Sciences

Closed book examination   Time: $2\frac{1}{2}$ hours

Name: ____________________ **Signature:** ____________________

Student Number: ____________________ **Section:** ____________________

No calculators, notes or other aids. You must show your work to obtain full credit. Express answers in terms of fractions or constants such as $\sqrt{3}$ or $\ln(4)$ rather than decimals. The last page contains some helpful formulae.

Problem	total possible	score
1.	18	
2.	6	
3.	12	
4.	7	
5.	9	
6.	10	
7.	15	
8.	10	
9.	6	
10.	7	
total	100	

Section 101 (MWF 10:00): J. Macdonald
Section 102 (MWF 8:00): J. Allard
Sections 103 (MWF 11:00) and 104 (MWF 1:00): Y-X. Li
Section 105 (TuTh 9:30): R. Israel
Section 106 (MWF 8:00): A. Duncan

1. For this short-answer question, only the answers (placed in the boxes) will be marked.

(3 points) (a) Find the average rate of change of $f(x) = x^2/(1+x^2)$ on the interval $0 \leq x \leq 2$.

(3 points) (b) Find the global (absolute) minimum of the function $f(x) = x^2 e^{-x}$ on the interval $-3 \leq x \leq 3$.

(3 points) (c) Find the y-intercept of the tangent line at the point $(1, -1)$ to the graph of the function $y(x)$ defined implicitly by $2x^3 + y^3 + xy = 0$.

(3 points) (d) Which of the following differential equations has $y(t) = 4e^{2t}$ as a solution?
(i) $y' = -4y$ **(ii)** $y' = -2y$
(iii) $y'' = 4y$ **(iv)** $y'' = -4y$

(3 points) (e) Given the curve $y = Vx^2/(x^2 + K^2)$, find new variables $X = X(x)$ and $Y = Y(y)$ so that the graph of Y as a function of X is a straight line.

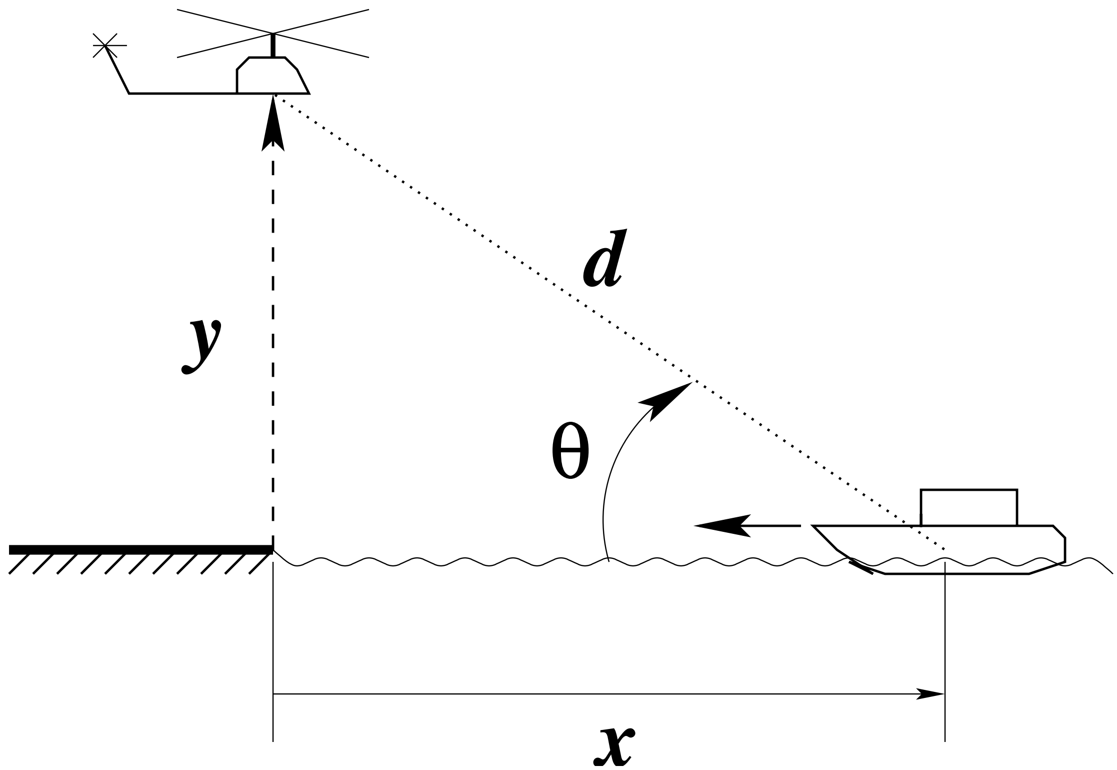
(3 points) (f) Find an approximate value for $\cos(\pi/3 + \sqrt{3}/100)$, based on known values of functions at $x = \pi/3$.

(6 points) 2. Do only **one** of the following two questions. If you write something for both, circle (a) or (b) to indicate which one you want marked.

(a) Using the definition of the derivative (no differentiation rules), find $f'(1)$ where $f(x) = (4x + 12)^{1/2}$.

Hint: $a^{1/2} - b^{1/2} = \dfrac{a - b}{a^{1/2} + b^{1/2}}$

(b) Consider the differential equation $\dfrac{dy}{dt} = \dfrac{y^2 - 1}{y^2 + 1}$ with the initial condition $y(0) = 0$. Use two steps of Euler's method to find an approximate value for $y(1)$.



3. A navy vessel is located at 25 km from the enemy shoreline where an enemy helicopter base is located. The vessel is moving toward the base at a constant speed of $v_v = 30\ km/h$. At this time (referred to as $t = 0$ for convenience), its early warning radar detected that an enemy helicopter is located on the ground in the base and is taking off vertically with a constant speed $v_h = 40\ km/h$ (see figure). To determine the optimal time to fire a laser beam to shoot down the helicopter, the following information must be provided to the commander.

(6 points) (a) At what time will the distance between the vessel and the helicopter be shortest? What is this distance? Assume that the speeds and directions of both the vessel and helicopter are constant.

$t =$ $\boxed{}$ hours, distance = $\boxed{}$ km

(6 points) (b) The laser is kept pointing directly at the helicopter at all times. At the time found in (a), what is the rate of change of the angle θ at which the laser is aimed (see figure)?

$$\frac{d\theta}{dt} = \boxed{} \text{ radians}/h$$

(7 points) 4. Find all points on the graph of $y^2 = 4x + 12$ where the tangent line to that graph is parallel to $y = x + 1$.

5. A model rocket moves vertically so that its height (in metres) above the ground at time t seconds is $y(t) = 900t - t^3$ from $t = 0$ until the rocket hits the ground.

(3 points) (a) When does the rocket reach its greatest height? Call this time t_1.

$t_1 = \boxed{}$ seconds

(3 points) (b) When does the rocket hit the ground? Call this time t_2.

$t_2 = \boxed{}$ seconds

(3 points) (c) What is the rocket's average velocity from time t_1 to time t_2?

$\overline{v} = \boxed{}$ m/s

6. Newton's Law of Cooling states that the rate of change of the temperature of an object is proportional to the difference in temperature between the environment and the object. In a room at 20° C, suppose a cup of coffee cools from 80° C to 60° C in 5 minutes.

(5 points) (a) How long does it take to cool from 80° C to 30° C?

It takes $\boxed{}$ minutes.

(5 points) (b) What is its temperature 15 minutes after it was at 75° C?

Temperature = $\boxed{}$ ° C.

7. Consider the function

$$f(x) = 10\ln(x^2+1) - x^2$$

(7 points) (a) Find all the critical points of $f(x)$ and classify each one as a local maximum, local minimum or neither.

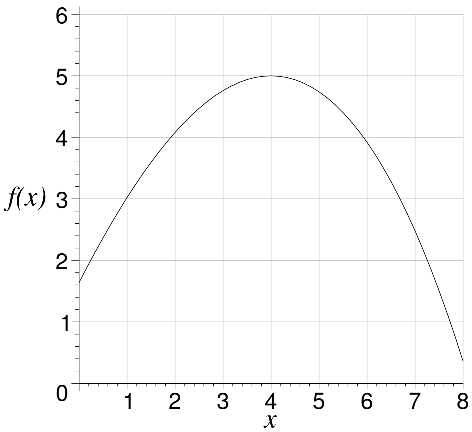
(4 points) (b) Does this function have a global maximum on the interval $(-\infty, \infty)$? Does it have a global minimum there? Circle Yes or No in each case.

Global maximum? Yes No Global minimum? Yes No

(4 points) (c) $f(1)$ is close to 6. Suppose we use Newton's method to find the solution of $f(x) = 6$, starting with the initial guess $x_0 = 1$. Find x_1.

$x_1 = \boxed{}$

8. The graph of a function $f(x)$ for $0 \leq x \leq 8$ is shown below.



(2 points) (a) What is $f(4)$?

$f(4) =$ ______

(2 points) (b) What is $f'(4)$?

$f'(4) =$ ______

(2 points) (c) What is the sign of $f''(4)$ (positive, negative or zero)?

$f''(4)$ is ______

(4 points) (d) $f(x)$ is actually a polynomial. If $f''(-4) < 0$, what can you say about the number of inflection points between $x = -4$ and $x = 4$?

The number is ______

(6 points) 9. Two quantities x and y, both depending on time t, are related by the equation $x^2 - y^3 = 1$. If $dx/dt = 4$, what is dy/dt at the moment when $x = 3$ and $y = 2$?

$$\frac{dy}{dt} = \boxed{}$$

10. Consider the differential equation

$$\frac{dy}{dx} = \frac{y^2 - 1}{y^2 + 1}$$

(4 points) (a) Find all stable and unstable steady states (equilibrium values), if any, for this differential equation.

Stable : $\boxed{}$ Unstable: $\boxed{}$

(3 points) (b) What value will the solution of this differential equation with initial condition $y(0) = 0$ approach as t increases?

$$\lim_{t \to +\infty} y(t) = \boxed{}$$

Useful Formulae

Law of cosines:

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

Trig identities:

$$\begin{aligned}
&\sin^2\theta + \cos^2\theta = 1\\
&\sin(A+B) = \sin A\cos B + \cos A\sin B\\
&\cos(A+B) = \cos A\cos B - \sin A\sin B\\
&\tan\theta = \frac{\sin\theta}{\cos\theta}
\end{aligned}$$

Values:

θ	$\sin\theta$	$\cos\theta$
0	0	1
$\pi/6$	$1/2$	$\sqrt{3}/2$
$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$
$\pi/3$	$\sqrt{3}/2$	$1/2$
$\pi/2$	1	0