Math 101—Final Examination
Duration: 150 minutes

Surname (Last Name) ____________ Given Name ____________ Student Number ____________

Do not open this test until instructed to do so! This exam should have 12 pages, including this cover sheet. No textbooks, notes, calculators, or other aids are allowed; phones, pencil cases, and other extraneous items cannot be on your desk. Turn off cell phones and anything that could make noise during the exam. Phones cannot be visible at any point during the exam.

UBC rules governing examinations:

1. Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
2. Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
3. No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no examination candidate shall be permitted to enter the examination room once the examination has begun.
4. Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
5. Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
   (a) speaking or communicating with other examination candidates, unless otherwise authorized;
   (b) purposely exposing written papers to the view of other examination candidates or imaging devices;
   (c) purposely viewing the written papers of other examination candidates;
   (d) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
   (e) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)—(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
6. Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
7. Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
8. Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

Problem	**Out of**	**Score**	**Problem**	**Out of**	**Score**
1	8		7	6	
2	8		8	7	
3	6		9	7	
4	6		10	7	
5	6		11	8	
6	6		**Total**	75	

Problems 1–2 are answer-only questions: the correct answers in the given boxes earns full credit, and no partial credit is given.

1a. **[2 pts]** Only one of the following statements is always true; determine which one is true.

(Assume $f(x)$ and $g(x)$ are continuous functions.) Answer: $\boxed{}$

A: $\displaystyle\int_{-3}^{-2} f(x)dx = -\int_{3}^{2} f(x)dx.$

B: $\displaystyle\int_{0}^{1} f(x)\cdot g(x)\,dx = \int_{0}^{1} f(x)\,dx \cdot \int_{0}^{1} g(x)\,dx.$

C: If $\displaystyle\int_{1}^{\infty} f(x)\,dx$ converges and $g(x) \geq f(x) \geq 0$ for all x, then $\displaystyle\int_{1}^{\infty} g(x)\,dx$ converges.

D: If $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}b_n$ converges, then $\displaystyle\sum_{n=1}^{\infty} b_n$ also converges.

E: When $f(x)$ is positive and concave up, any Trapezoid Rule approximation for $\displaystyle\int_{a}^{b} f(x)\,dx$ will be an upper estimate for $\displaystyle\int_{a}^{b} f(x)\,dx.$

1b. **[2 pts]** For each integral, choose the substitution type that is most beneficial for evaluating the integral. (Write **F**, **G**, or **H** in each box; each answer will be used exactly once.)

F: $x = a\sec\theta$ **G:** $x = a\sin\theta$ **H:** $x = a\tan\theta$

(i) $\displaystyle\int \frac{2x^2}{\sqrt{9x^2-16}}\,dx$ Answer: $\boxed{}$

(ii) $\displaystyle\int \frac{x^4-3}{\sqrt{1-4x^2}}\,dx$ Answer: $\boxed{}$

(iii) $\displaystyle\int (25+x^2)^{-5/2}\,dx$ Answer: $\boxed{}$

1c. **[2 pts]** For which values of the constant q does the integral $\displaystyle\int_{0}^{1} x^q\,dx$ converge?

Answer: $\boxed{}$

J: It converges exactly when $q < -1$.
K: It converges exactly when $q \leq -1$.
L: It converges exactly when $q > -1$.
M: It converges exactly when $q \geq -1$.
P: It converges for all q.

1d. **[2 pts]** Calculate $\displaystyle\int_{-7}^{7} \big(x^3 + \sin(2x) + 1\big)\,dx$. Simplify your answer completely.

Answer: $\boxed{}$

Problems 1–2 are answer-only questions: the correct answers in the given boxes earns full credit, and no partial credit is given.

2a. **[2 pts]** Which of the following integrals equals $\int_1^2 f(x^3)\,dx$? Answer: $\boxed{}$

$\mathbf{A}\colon \int_1^2 3u^2 f(u)\,du$	$\mathbf{D}\colon \int_1^2 \frac{f(u)}{3u^{2/3}}\,du$	$\mathbf{G}\colon \int_1^2 f(u)\,du$
$\mathbf{B}\colon \int_1^{\sqrt[3]{2}} 3u^2 f(u)\,du$	$\mathbf{E}\colon \int_1^{\sqrt[3]{2}} \frac{f(u)}{3u^{2/3}}\,du$	$\mathbf{H}\colon \int_1^{\sqrt[3]{2}} f(u)\,du$
$\mathbf{C}\colon \int_1^8 3u^2 f(u)\,du$	$\mathbf{F}\colon \int_1^8 \frac{f(u)}{3u^{2/3}}\,du$	$\mathbf{J}\colon \int_1^8 f(u)\,du$

2b. **[2 pts]** Which of the following Maclaurin series equals $\dfrac{x^4}{1-2x^3}$? Answer: $\boxed{}$

$\mathbf{K}\colon x^3 - 2x^7 + 4x^{11} - 8x^{15} + \cdots$	$\mathbf{Q}\colon x^4 - 2x^7 + 4x^{10} - 8x^{13} + \cdots$
$\mathbf{L}\colon x^3 - \frac{1}{2}x^7 + \frac{1}{4}x^{11} - \frac{1}{8}x^{15} + \cdots$	$\mathbf{S}\colon x^4 - \frac{1}{2}x^7 + \frac{1}{4}x^{10} - \frac{1}{8}x^{13} + \cdots$
$\mathbf{M}\colon x^3 + 2x^7 + 4x^{11} + 8x^{15} + \cdots$	$\mathbf{T}\colon x^4 + 2x^7 + 4x^{10} + 8x^{13} + \cdots$
$\mathbf{P}\colon x^3 + \frac{1}{2}x^7 + \frac{1}{4}x^{11} + \frac{1}{8}x^{15} + \cdots$	$\mathbf{U}\colon x^4 + \frac{1}{2}x^7 + \frac{1}{4}x^{10} + \frac{1}{8}x^{13} + \cdots$

2c. **[4 pts]** For each of the following series, choose the appropriate statement. (Write the appropriate two-letter answer in each box; each answer will be used at most once.)

CI: Converges by Integral Test with $\int_1^\infty \frac{\ln x}{x}\,dx$
CT: Converges by Test for Divergence
CC: Converges by Comparison Test with $\sum_{n=1}^\infty \frac{1}{n^3}$
CA: Converges by Alternating Series Test
CL: Converges by Limit Comparison Test with $\sum_{n=1}^\infty \frac{1}{n^2}$
DI: Diverges by Integral Test with $\int_1^\infty \frac{\ln x}{x}\,dx$
DT: Diverges by Test for Divergence
DC: Diverges by Comparison Test with $\sum_{n=1}^\infty \frac{1}{\sqrt{n}}$
DA: Diverges by Alternating Series Test
DL: Diverges by Limit Comparison Test with $\sum_{n=1}^\infty \frac{\pi}{n^2}$

(i) $\displaystyle\sum_{n=1}^\infty \frac{\ln n}{n}$ Answer: $\boxed{}$

(ii) $\displaystyle\sum_{n=1}^\infty \frac{4n-1}{(2n^2+1)^2}$ Answer: $\boxed{}$

(iii) $\displaystyle\sum_{n=1}^\infty (-1)^n n$ Answer: $\boxed{}$

(iv) $\displaystyle\sum_{n=1}^\infty \frac{\arctan n}{n^2}$ Answer: $\boxed{}$

Problems 3–6 are short-answer questions: put a box around your final answer, but you must include the correct accompanying work to receive full credit.

3a. **[3 pts]** Calculate the area of the region enclosed by $y = 2^x$ and $y = \sqrt{x} + 1$.

3b. **[3 pts]** Evaluate $\displaystyle\int \tan^3 x \sec^5 x \, dx$.

Problems 3–6 are short-answer questions: put a box around your final answer, but you must include the correct accompanying work to receive full credit.

4a. **[3 pts]** Find the average value of the function $y = x^2 \ln x$ on the interval $1 \leq x \leq e$.

4b. **[3 pts]** Calculate $\displaystyle\int \frac{1}{x^4 + x^2}\, dx$.

Problems 3–6 are short-answer questions: put a box around your final answer, but you must include the correct accompanying work to receive full credit.

5a. **[3 pts]** Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{9n+5}$ is absolutely convergent, conditionally convergent, or divergent; justify your answer.

5b. **[3 pts]** The series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n+1)^2}$ converges to some number S (you don't have to prove this). According to the Alternating Series Estimation Theorem, what is the smallest value of n for which the nth partial sum of the series is at most $\frac{1}{100}$ away from S? For this value of n, write out the nth partial sum of the series.

Problems 3–6 are short-answer questions: put a box around your final answer, but you must include the correct accompanying work to receive full credit.

6a. **[3 pts]** Find the Maclaurin series for $\int x^4 \arctan(2x)\, dx$.

6b. **[3 pts]** Suppose that you have a sequence $\{b_n\}$ such that the series $\sum_{n=0}^{\infty}(1 - b_n)$ converges. Using the tests we've learned in class, prove that the radius of convergence of the power series $\sum_{n=0}^{\infty} b_n x^n$ is equal to 1.

Problems 7–11 are long-answer: give complete arguments and explanations for all your calculations—answers without justifications will not be marked.

7. Define $f(x) = x^3 \int_0^{x^3+1} e^{t^3}\,dt$.

(a) **[3 pts]** Find a formula for the derivative $f'(x)$.

(b) **[3 pts]** Find the equation of the tangent line to the graph of $y = f(x)$ at $x = -1$. Simplify your answer completely.

8. Let D be the region below the graph of the curve $y = \sqrt{9 - 4x^2}$ and above the x-axis.

(a) **[4 pts]** Using an appropriate integral, find the *area* of the region D; simplify your answer completely.

(b) **[3 pts]** Find the *centre of mass* of the region D; simplify your answer completely. (Assume it has constant density ρ.)

9.

(a) **[3 pts]** Find the solution of the differential equation $\dfrac{1+\sqrt{y^2-4}}{\tan x}y' = \dfrac{\sec x}{y}$ that satisfies $y(0)=2$. You don't have to solve for y in terms of x.

(b) **[4 pts]** A sculpture, shaped like a pyramid 3 m high sitting on the ground, has been made by stacking smaller and smaller iron plates on top of one another. The iron plate at height z m above ground level is a square whose side length is $(3-z)$ m. All of the iron plates started on the floor of a basement 2 m below ground level.

Write down an integral that represents the work, in newtons, it took to move all of the iron from its starting position to its present position. Do **not** evaluate the integral. (You can use $9.8\ \mathrm{m/s^2}$ for the force of gravity and $8000\ \mathrm{kg/m^3}$ for the density of iron.)

10. Let R be the region inside the circle $x^2 + (y-2)^2 = 1$. Let S be the solid obtained by rotating R about the x-axis.

(a) **[4 pts]** Write down an integral representing the volume of S.

(b) **[3 pts]** Evaluate the integral you wrote down in part (a). (You may use any technique you know, including a geometric one, to do so.)

11.

(a) **[4 pts]** Find the *value* of the convergent series

$$\sum_{n=2}^{\infty}\left(\frac{2^{n+1}}{3^n}+\frac{1}{2n-1}-\frac{1}{2n+1}\right)$$

(simplify your answer completely).

(b) **[4 pts]** Find the *interval of convergence* for the power series

$$\sum_{n=1}^{\infty}\frac{(x-2)^n}{n^{4/5}(5^n-4)}.$$