The University of British Columbia

Sessional Examinations - April 2011

MATH 101

Integral Calculus with Applications to Physical Sciences and Engineering

Closed book examination Time: 2.5 hours

Last Name: ____________________ **First Name:** ____________________

Student Number: ____________________ **Instructor's Name:** ____________________

Signature: ____________________ **Section Number:** ____________________

Special Instructions:

No books, notes, or calculators are allowed.

Rules governing examinations

- Each candidate must be prepared to produce, upon request, a UBC-card for identification.
- Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
- No candidate shall be permitted to enter the examination room after the expiration of one half hour from the scheduled starting time, or to leave during the first half hour of the examination.
- Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.
  (a) Having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners.
  (b) Speaking or communicating with other candidates.
  (c) Purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received.
- Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.
- Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

1		30
2		8
3		10
4		24
5		8
6		8
7		4
8		8
Total		100

[30] **1. Short-Answer Problems**. Put your answer in the box provided but show your work also. Each question is worth 3 marks, but not all questions are of equal difficulty. Full marks will be given for correct answers placed in the box, but at most 1 mark will be given for incorrect answers. **Simplify your answers as much as possible in this question.**

(a) Evaluate the definite integral:

$$\int_{-1}^{0} x^2\sqrt{1+x^3}\,dx.$$

Answer:

(b) Find the derivative $f'(t)$, if

$$f(t) = \int_{t^4}^{2} \sqrt{1+x^3}\,dx.$$

Answer:

(c) Find the average value of the function on the given interval:

$$f(x) = xe^x, \quad [0,2].$$

Answer:

(d) A tank in the shape of a cube with edges 2 ft long is filled with a fluid that weighs 50 lb/ft^3. Find the hydrostatic force, in lb, against one of the vertical sides of the tank.

Answer:

(e) Determine whether the sequence is convergent or divergent. In case it is convergent, compute its limit:

$$\left\{ \frac{\sin(n)}{\ln(n)} \right\}_{n=2}^{\infty}.$$

Answer:

(f) Determine whether the series is convergent or divergent. In case it is convergent, compute its sum:

$$\sum_{n=1}^{\infty} \frac{7 \cdot 3^{2n-1}}{10^n}.$$

Answer:

(g) How many terms of the convergent series $\sum_{n=1}^{\infty} n^{-3}$ are needed to ensure that the sum is accurate to within $\frac{1}{20\,000}$?

Answer:

(h) Find all real numbers p so that the series is convergent:

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^p} = 1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \dots .$$

Answer:

(i) Determine whether the series is absolutely convergent, conditionally convergent, or divergent:

$$\sum_{n=0}^{\infty} \frac{(-2011)^n}{n!}.$$

Answer:

(j) Find the first three nonzero terms of the Maclaurin series (power series in x) for

$$f(x) = x^3 \sin(x^3).$$

Answer:

Full-Solution Problems. In questions 2–8, justify your answers and **show all your work**. If a box is provided, write your final answer there. Unless otherwise indicated, **simplification of answers is not required in these questions**.

[8] **2**. (a) Find the total area of the finite plane region lying between the curves $y = x$ and $y = x^3$.

(b) Consider a solid whose base in the xy-plane is the finite region bounded by the curves $y = x^2$ and $y = 2 - x^2$. The cross sections of the solid perpendicular to the x-axis with one side in the xy-plane are squares. Find the volume of this solid.

[10] **3**. (a) A tank in the shape of a vertical circular cylinder 2 m high and with radius 0.5 m is filled with a fluid that has density 1 000 kg/m^3. Write a definite integral that gives the work, in Joules, required to pump all the fluid to the level of the top of the tank. For acceleration due to gravity, use $g = 9.8$ m/s^2. **Do not evaluate the integral.**

(b) Find the centroid of the finite region in the plane bounded by the x-axis and the curve $y = 1 + \sin x$, for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.

(Blank page for calculations.)

[24] **4.** (a) Evaluate the integral (*hint:* put $u = \sqrt{x}$)

$$\int \frac{\sqrt{x}}{x-1}\, dx.$$

(b) Evaluate the integral

$$\int x^3 \sqrt{1+x^2}\, dx.$$

(c) Determine whether the following improper integral is convergent or divergent. Remember to justify your answer:

$$\int_1^\infty \frac{\sqrt{x}}{x^2+x}\,dx.$$

(d) Determine whether the following series is convergent or divergent. Remember to justify your answer:

$$\sum_{n=2}^{\infty}\left(\frac{1}{\ln n}\right)^2 \frac{1}{n}.$$

[8] **5**. Bird-Bath and Beyond Incorporated is famous for its Quetzal attracting bird-feeder solution made from water, honey and cane-sugar.
To make their solution, both honey and a cane-sugar solution are poured into a 200 ℓ mixing tank. The honey is poured in at a rate of 1 ℓ per minute while the sugar solution is poured in at 9 ℓ per minute. Note that 1 ℓ of honey contains 1 kg of sugar, while the cane-sugar solution contains 100 g of sugar per ℓ.
Unfortunately today there is a problem with the mixing tank. It was thoroughly cleaned and is initially filled with pure water, but main valve was broken and the water cannot be drained. When the mixing process is started, the honey and sugar-solutions are poured into the tank, and the excess fluid flows out of an emergency valve at 10 ℓ per minute and onto the floor.
You should assume that the solutions mix immediately and thoroughly in the tank.

(a) Give a differential equation that is satisfied by the amount of sugar in the tank after time t.

(b) How much sugar is in the tank after time t?

(c) The bird feeder solution can only be sold if there is at least 30 kg of sugar in the 200 ℓ tank. If the flow of sugar solution and honey is switched off after 20 minutes, can the result be sold?

(Blank page for calculations.)

[8] **6**. (a) Evaluate the following indefinite integral as a power series, and find the radius of convergence:

$$\int \frac{1}{1+x^3}\,dx.$$

(b) Use part (a) to determine an approximation of the following definite integral so that the error in the approximation satisfies $|\text{error}| \leq 10^{-9}$. Remember to justify your answer:

$$\int_0^{0.1} \frac{1}{1+x^3}\,dx.$$

[4] **7**. Evaluate

$$\int_{-1}^{2} (x-1)\,dx$$

using a limit of Riemann sums. **No credit will be given for using another method** (but you can use another method to check your answer). You may use the formulas

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \text{and} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

[6] **8**. Find (a) the radius of convergence, and (b) the interval of convergence of the following power series, carefully justifying your answer:

$$\sum_{n=0}^{\infty} \frac{(x-2)^n}{\ln(n+2)}.$$

The End