MATH 101B — Final Examination — 150 minutes

22 April 2024

- The test has 30 pages, with 20 questions worth a total of 90 marks.
- **A formula sheet is provided on the last page.**
- This is a closed-book examination. **None of the following are allowed**: documents, formula sheets, electronic devices of any kind (including calculators, cell phones, ...).
- No work on this page will be marked.

Student number								
Section								
Name	....							
Signature								

Look at both sides of every sheet.
Some questions start on the back.

Turn off and put away all cell phones, pagers, alarms, etc., before the exam begins. Any such device that disrupts the exam will be confiscated.

Rules Governing Formal Examinations

1. Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, their UBCcard for identification.

2. Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.

3. No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no examination candidate shall be permitted to enter the examination room once the examination has begun.

4. Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.

5. Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:

   i. speaking or communicating with other examination candidates, unless otherwise authorized;
   ii. purposely exposing written papers to the view of other examination candidates or imaging devices;
   iii. purposely viewing the written papers of other examination candidates;
   iv. using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
   v. using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)—(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).

6. Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.

7. Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

Additional rules governing this examination:

1. For every question, write your answers in *the space provided for that question.*
2. Use both sides of every page.
3. If you need more work space, request an official Extra Sheet from an invigilator. On any Extra Sheet, write your identification on every side you want marked. (You may use different sides for different questions.)
4. Work on this page will not be marked.
5. This is a closed-book exam.
   (a) Calculators and other calculating devices may not be used.
   (b) Notes may not be used.
   (c) Watches must be removed and taken off the table.
   (d) Phones must be turned off and stored in an inaccessible location (like inside a backpack).
6. If an answer box is provided, you must write your answer in the box. (The justification may appear outside the box.)
7. You must justify your answers unless an explicit exception is made.
8. You may quote, and then use without proof, any result proven in class or on assignments.
9. Grading will account for well known presentation standards, including but not limited to the following.
   (a) Correct mathematical notation is required throughout.
   (b) Answers must be simplified unless an explicit exception is made.
      - For example, write $\log\left(e^{\sqrt{2}}\right) = \sqrt{2}$, but do not write $\sqrt{2} \approx 1.414$.
      - Any answer of the form "trig (arctrig)" (for example, $\sin(2 \arctan x)$) is considered inadequately simplified and will not receive full marks.
      - If a question expressly allows you to write a final answer in "calculator-ready form," you should write something that could easily be evaluated using a basic scientific calculator. Obvious simplifications that demonstrate basic understanding, such as "$\sin \pi = 0$" or "$7^0 = 1$," must *still* be performed for full marks.
   (c) The bounds on definite integrals must correspond to the variable in the differential. (A differential must be present in all integrals.)
   (d) Improper integrals must be recognized as such, and evaluated with explicit reference to limits.

Part I: 15 questions, each worth 3 marks

(3 pt) 1. Find $F'(x)$, given

$$F(x) = \int_0^{x^2} \frac{4-t}{1+\cos^2(t)}\,\mathrm{d}t.$$

(3 pt) 2. Find the area of the region under the curve $y = 8 - 2x^2$ and above the x-axis.

(3 pt) 3. Evaluate $\int \sqrt{9 - x^2}\, \mathrm{d}x$.

(3 pt) 4. Evaluate $\int_{-x^2}^{x^2} t^2 \cos(t^3)\,\mathrm{d}t$.

(3 pt) 5. Evaluate $\int_0^{\pi/2} \sin^4 x \cos^3 x \; \mathrm{d}x$.

(3 pt) 6. Evaluate $\int \arctan(2x)\, \mathrm{d}x$.

(3 pt) 7. Find the area of the region enclosed between $y = \sin x$ and $y = \cos x$ from $x = 0$ to $x = \pi$.

(3 pt) 8. Find the general solution $y(t)$ of the differential equation $\dfrac{\mathrm{d}y}{\mathrm{d}t} = te^{-y}$.

(3 pt) 9. Decide whether the following improper integral converges or diverges:

$$\int_1^\infty \frac{x^2+1}{x^3+2}\,\mathrm{d}x.$$

(Note that the *value* of this integral is not requested.)

(3 pt) 10. Evaluate the following improper integral, or show that it diverges:

$$\int_0^1 \frac{\mathrm{d}x}{(1-x)^{3/2}}.$$

(3 pt) 11. How many intervals are needed to guarantee that the Simpson's Rule approximation of the integral

$$I = \int_0^5 f(x)\,\mathrm{d}x$$

lies within 0.0004 of the exact value? Assume that

$$\left|f^{(4)}(x)\right| \le 9 \qquad \text{for all } x \text{ obeying } 0 \le x \le 5,$$

and note that $0.0004 = \dfrac{1}{20 \cdot 5^3}$.

(3 pt) 12. Let θ_n be a sequence such that $0 < \theta_n < 3$ for all n and

$$\sum_{n=1}^{\infty} \cos(\theta_n) = 2.$$

Find $\lim_{n\to\infty} \theta_n$, or show that the limit does not exist.

(3 pt) 13. Find the set of all $q > 0$ for which the following series converges:

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{(3n)^q - 1}}.$$

As in all questions, remember to fully justify your answer.

(3 pt) 14. Find the radius of convergence of the power series

$$\sum_{n=0}^{\infty}(3^n + n^3)x^n.$$

(3 pt) 15. Find $\int \frac{1}{(x+1)(x-3)} \, \mathrm{d}x$.

Part II: 5 questions, each worth 9 marks

16. ★★★☆ Consider the separable differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} - 2xy + 4y = 8 - 4x, \qquad -\infty < x < +\infty.$$

(5 pt) (a) Find the general solution. Express your answer in the form $y = f(x)$.

(4 pt) (b) A certain solution satisfies $y(0) = 8$. For this solution, what is the smallest value of $y(x)$ over all real x?

Extra space reserved for Question 16

17. ★★☆☆ A random variable X has the following probability density function (PDF):

$$f(x) = \begin{cases} 1, & \text{for } -\dfrac{1}{2} < x < 0, \\ 2, & \text{for } 0 < x < c, \\ 0, & \text{otherwise.} \end{cases}$$

(1 pt) (a) Find c.

(1 pt) (b) Find $Pr\left(-1 \leq X \leq -\frac{1}{4}\right)$.

(3 pt) (c) Find $\mathbb{E}(X)$.

(4 pt) (d) Find all values of X whose distance from $\mathbb{E}(X)$ is at most one standard deviation. Write your answer as an interval.

For this part only, your endpoints may be left in calculator-ready form.

Extra space reserved for Question 17

(9 pt) 18. ★★★☆ A key statistic for describing how a population changes is the *average population* over a fixed time interval, such as a season, a year, or a generation.

For any given function f, the *average value* from $t = a$ to $t = b$ is

$$\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t.$$

Suppose the population (in thousands) of a species at time t (in months) is given by the function

$$P(t) = 1 + e^{-(t-4)^2} + \cos(\pi t).$$

For which four-month period (i.e., from $a = T$ to $b = T + 4$) is *the average value of* P the greatest? You need to justify your answer.

Extra space reserved for Question 18

(9 pt) 19. ★★★☆ For a constant $a > 0$, consider the plane region lying above the curve $y = a^2x^2$ and below the curve $y = \sqrt{ax}$. Rotating this region around the x-axis produces a solid: let V_x denote the resulting volume. Similarly, let V_y denote the volume produced by rotating the same region around the y-axis.

Find the value of a for which $V_y = 2V_x$.

Extra space reserved for Question 19

20. ★★★★ Let $f(x) = \int_0^x te^{-t}\, dt$. Use this function in all parts below.

(3 pt) (a) Solve the integral to produce a simple formula for $f(x)$.

(2 pt) (b) Given that the following series converges, find the constant c:

$$S = \sum_{n=0}^{\infty} (c - f(n)) .$$

(4 pt) (c) For the constant c found above, find the exact value of S.

Extra space reserved for Question 20

Blank page for rough work. Nothing on this side of the page will be marked!

Blank page for rough work. Nothing on this side of the page will be marked!
It's OK to separate this page from the others, but you must hand it in anyway.

Formulas

Trigonometric Identities:

$$\cos^2(x) = \frac{1+\cos(2x)}{2}, \qquad \cos(2x) = \cos^2(x) - \sin^2(x), \qquad \sin^2(x) + \cos^2(x) = 1,$$

$$\sin^2(x) = \frac{1-\cos(2x)}{2}, \qquad \sin(2x) = 2\sin(x)\cos(x), \qquad \tan^2(x) + 1 = \sec^2(x).$$

Two Standard Integrals:

$$\int \tan(x)\,dx = \log|\sec(x)| + C\,, \qquad \int \sec(x)\,dx = \log|\sec(x) + \tan(x)| + C.$$

Taylor polynomial error bound: If $|f^{(n+1)}(c)| \le M$ for all c between a and x, then

$$|T_n(x) - f(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}.$$

Trapezoidal rule error bound: If $|f''(x)| \le M$ for all $a \le x \le b$, then

$$\left| T_n - \int_a^b f(x)\,dx \right| \le \frac{M}{12}\frac{(b-a)^3}{n^2}.$$

Simpson's rule error bound: If $|f^{(4)}(x)| \le L$ for all $a \le x \le b$, then

$$\left| S_n - \int_a^b f(x)\,dx \right| \le \frac{L}{180}\frac{(b-a)^5}{n^4}.$$

Taylor series:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}\,, \qquad -\infty < x < \infty$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}\,, \qquad -\infty < x < \infty$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n}\,, \qquad -\infty < x < \infty$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n\,, \qquad -1 < x < 1$$

$$\log(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}x^{n+1}\,, \qquad -1 < x \le 1$$

$$\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}x^{2n+1}\,, \qquad -1 \le x \le 1$$

Nothing written on this side of the page will be marked.