MATH 100 Final Exam
Duration: 2.5 hours

Student number								
Section								
Name	....							
Signature								

This test contains 30 questions on 38 pages worth a total of 100 marks.
Please ensure that your test is complete.

Rules regarding paper handling

1. Do not remove pages, staples, or otherwise disassemble your test.
2. You may request scrap paper. An invigilator will bring it to you. If you do not wish for this page to be graded mark is as "scrap."
3. You must submit all scrap paper whether you intend for it to be graded or not. No paper may be removed from the exam room.
4. This exam contains lots of blank space for your work. If you require additional paper, an invigilator will bring it to you. If you would like work on this additional paper to be graded you must include on each **side** of each extra page: (a) your unique 4 digit test number (b) the question number (c) your name (d) your student number. In particular, your **test number**, (a), can be found at the top centre of this page. Failure to include all of (a)-(d) will result in your extra page not being graded. Do not include work for more than one question on a single side of an extra sheet.

Rules governing UBC examinations:

1. Each candidate must be prepared to produce, upon request, a UBC card for identification.
2. No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination.
3. Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action:
   (a) Having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners;
   (b) Speaking or communicating with other candidates;
   (c) Purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received.
4. Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.
5. Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

Additional rules governing this examination:

1. This is a closed-book exam.
   (a) Calculators and other calculating devices may not be used.
   (b) Notes may not be used.
   (c) Watches must be removed and taken off the table.
2. If an answer box is provided, you must write down your answer (but not its justification) in the box.
3. Answers must be simplified and calculator-ready. For example, write $\log\left(e^{\sqrt{2}}\right) = \sqrt{2}$, but do not write $\sqrt{2} \approx 1.414$.
4. You must justify your answers unless an explicit exception is made.
5. You may use any result proven in class or on assignments.

1. (★☆☆☆, 2 marks) List the vertical and horizontal asymptotes of $y = \dfrac{x-4}{\sqrt{4x^2+3}}$.

Leave the field blank if there is no such asymptote.

Vertical asymptote(s):

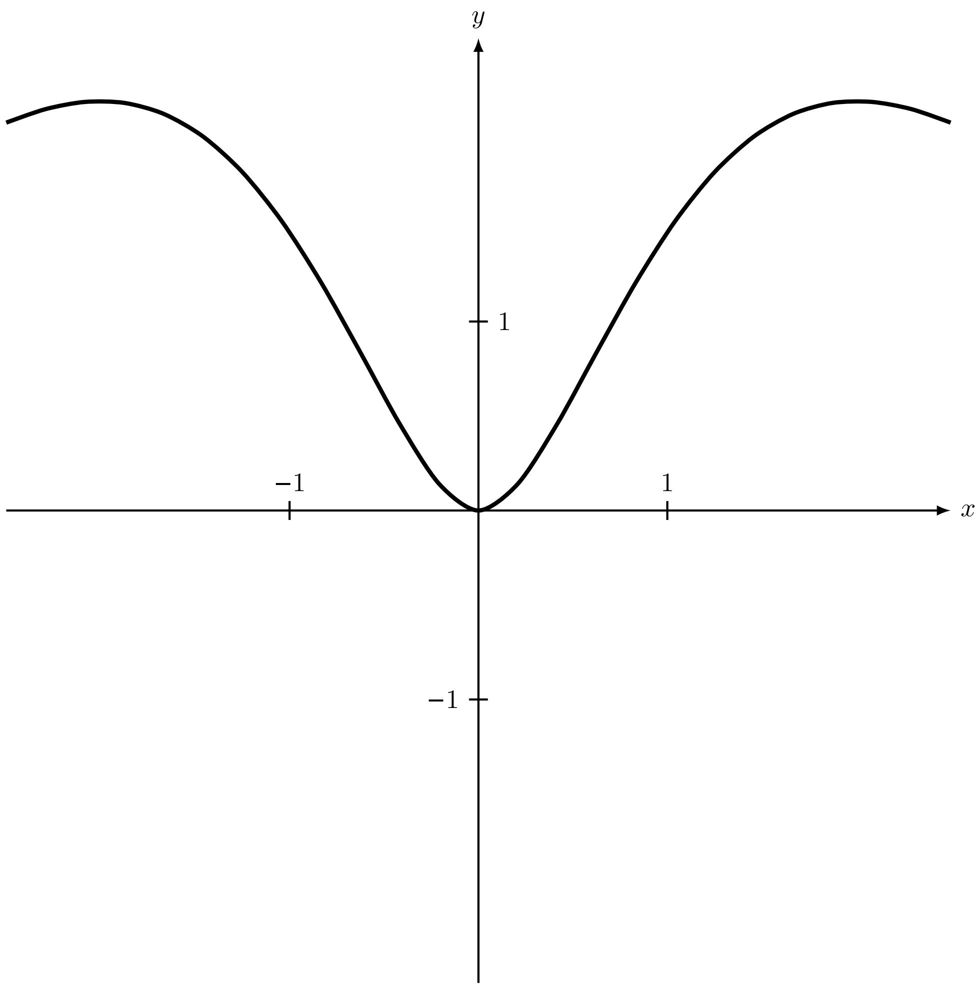
Horizontal asymptote(s):

2. (★☆☆☆, 2 marks) Find the value of the constant c that makes $f(x)$ continuous, where

$$f(x) = \begin{cases} cx + 3 & \text{if } x \leq 3 \\ cx^2 - 3 & \text{if } x > 3 \end{cases}.$$

Answer:

3. (★☆☆☆, 2 marks) A function $f(x)$ is pictured below. Sketch the graph of the derivative directly on the axes.



4. (★☆☆☆, 2 marks) Let $f(x) = -4e^{2x} + e^{-1}$. Find $f'(0)$.

Answer:

5. (★☆☆☆, 2 marks) Let $f(x) = \dfrac{1}{\sin(x) + \cos(x)}$. Find $f'\left(-\frac{\pi}{2}\right)$.

Answer:

6. (★☆☆☆, 2 marks) Let $f(x) = x^x$. Find $f'(e)$.

Answer:

7. (★☆☆☆, 2 marks) Let $f(x) = \sqrt{4 + 5x^4}$. Find $f'(-1)$.

Answer:

8. (★☆☆☆, 2 marks) Let

$$2x^2\left(f(x)\right)^2 = 0$$

and $f(3) = -1$. Find $f'(3)$.

Answer:

9. (★☆☆☆, 2 marks) Let $f(x) = \arctan(4x)$. Find $f'\left(\frac{1}{4}\right)$.

Answer:

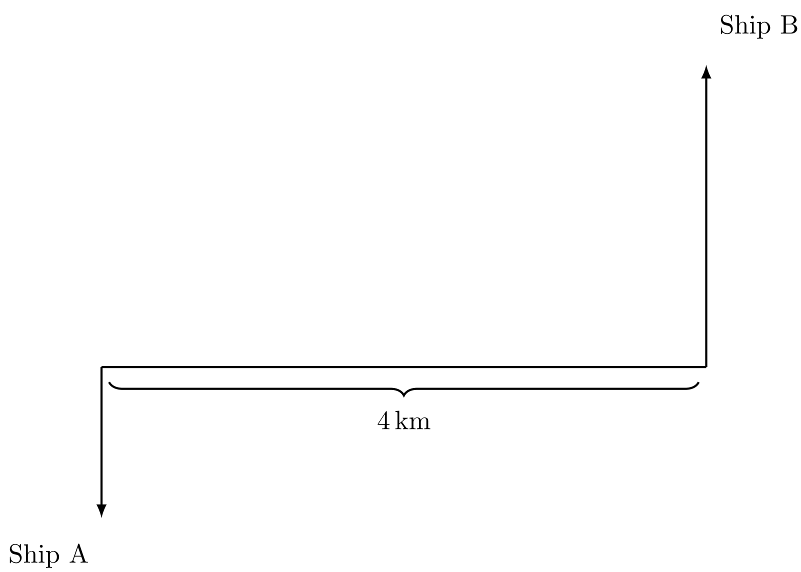
10. (★☆☆☆, 2 marks) Find the equation of the tangent line to the curve $y = x\sqrt{x}$ at the point $(1, 1)$. Your answer must be written in the form $y = mx + b$.

Answer:

11. (★☆☆☆, 2 marks) Let A be the area of a circle with radius r. If r changes at a rate of $2\,\text{cm/s}$, at what rate is the area enclosed by the circle changing when $r = 3\,\text{cm}$?

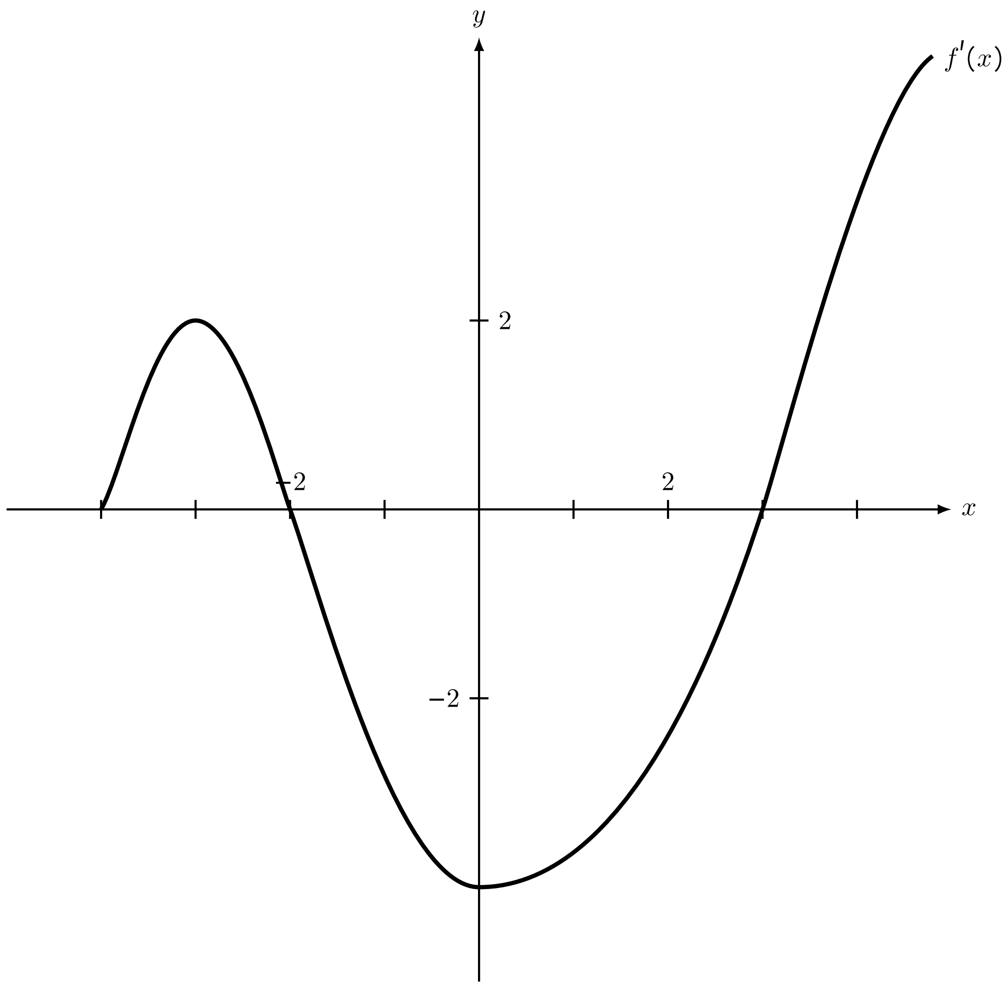
Answer:

12. (★☆☆☆, 2 marks) At noon, ship A is 4 km west of ship B. Ship A is sailing south at 2 km/h and ship B is sailing north at 4 km/h. The scenario some time after noon is pictured below. How fast is the distance between the ships changing 30 minutes after noon?



Answer:

13. (★☆☆☆, 2 marks) Consider the following graph of a *derivative* $f'(x)$.



Find all the intervals where $f(x)$ is concave down.

Answer:

14. (★☆☆☆, 2 marks) Let $f(x) = x^4 - 4x^3 - 18x^2 + 13$. Find all intervals where $f(x)$ is concave up.

Answer:

15. (★☆☆☆, 2 marks) Let

$$f(x) = x + \frac{4}{x}$$

be defined on the domain $x > 0$. Find the x-values of all extrema on that interval, indicating for each x-value found whether there is a local maximum or a local minimum there.

Answer:

16. (★☆☆☆, 2 marks) Use the linear approximation of $f(x) = \log(x)$ at $x = 1$ to estimate $\log(1.06)$.

Answer:

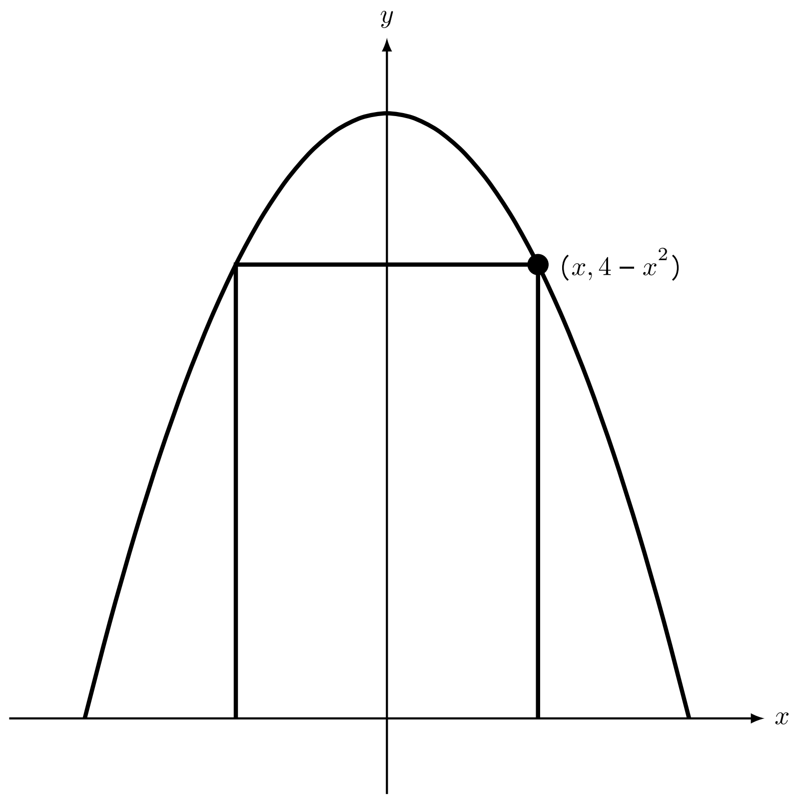
17. (★☆☆☆, 2 marks) Find the coefficient of the x^3 term of the Taylor polynomial of $f(x) = \dfrac{e^x}{1-x}$ about the point $x = 0$.

Answer:

18. (★☆☆☆, 2 marks) At what value of x on the curve $y = x + x^2 - x^3$ does the tangent line have the largest positive slope?

Answer:

19. (★☆☆☆, 2 marks) A rectangle is inscribed with its base on the x-axis and its upper corners on the parabola $y = 4 - x^2$, as shown below.



What is the height of such a rectangle with the greatest possible area?

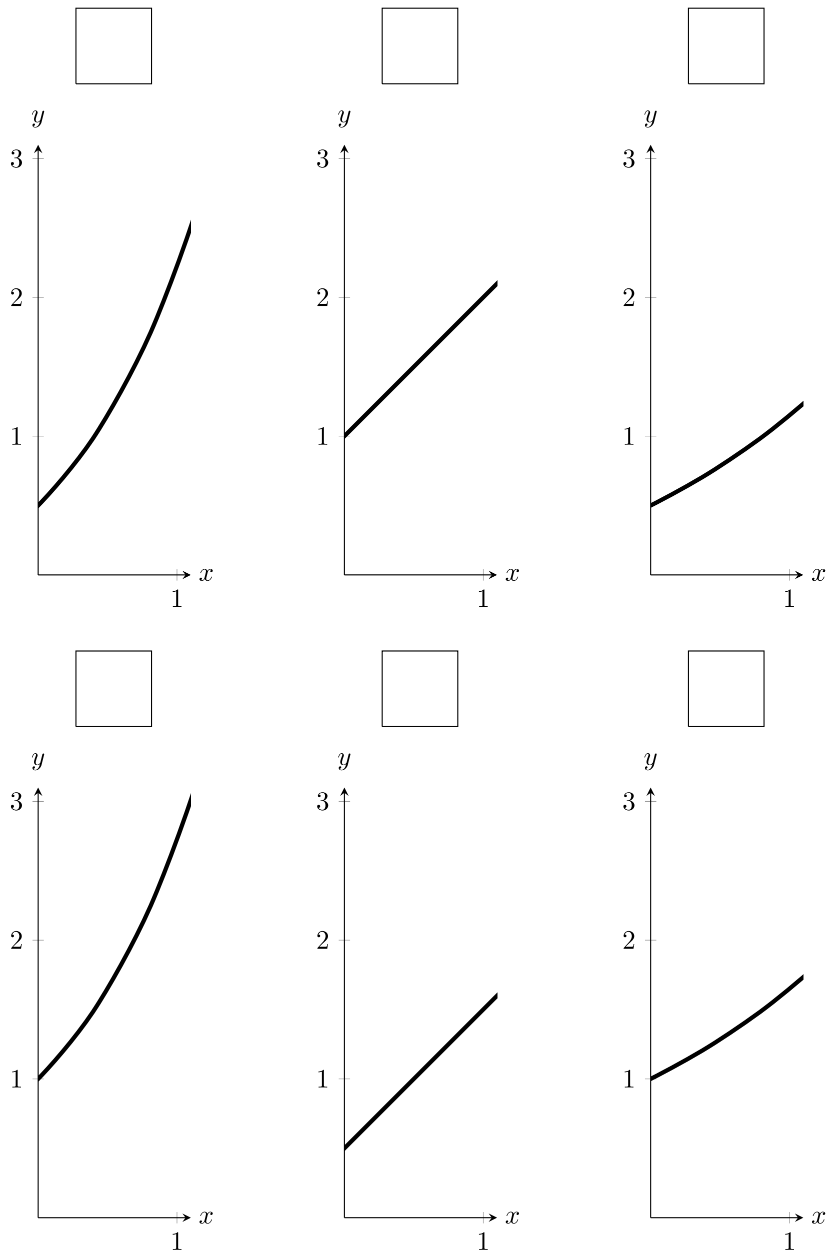
Answer:

20. (★☆☆☆, 2 marks) Find all values of r such that $y = e^{rt}$ solves the differential equation

$$y'' + 6y' + 5y = 0.$$

Answer:

21. (★☆☆☆, 2 marks) Which of the following is a solution to the differential equation $y' = \frac{y}{2}$ with initial condition $y(0) = 1$? Select all that apply.



22. (★☆☆☆, 2 marks) Sketch the phase line for the differential equation $x' = (x + 1)(x - 4)$ below. Clearly indicate the equilibrium points, and indicate with arrows whether x is increasing or decreasing on either side of each equilibrium point.

x

23. (★☆☆☆, 2 marks) Which of the following is the slope field for the differential equation $\frac{dy}{dx} = x + y$? Select all that apply.

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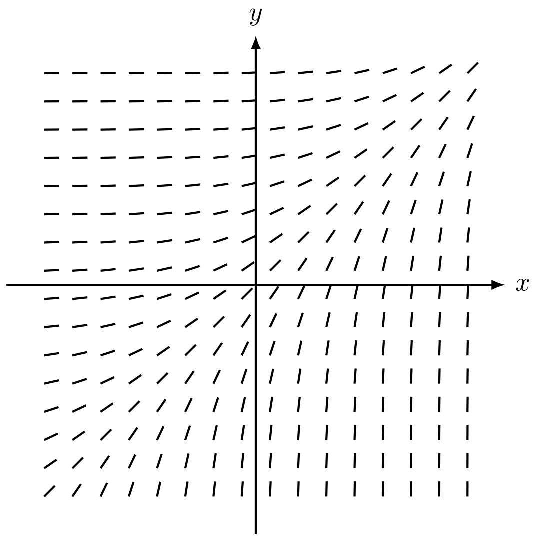

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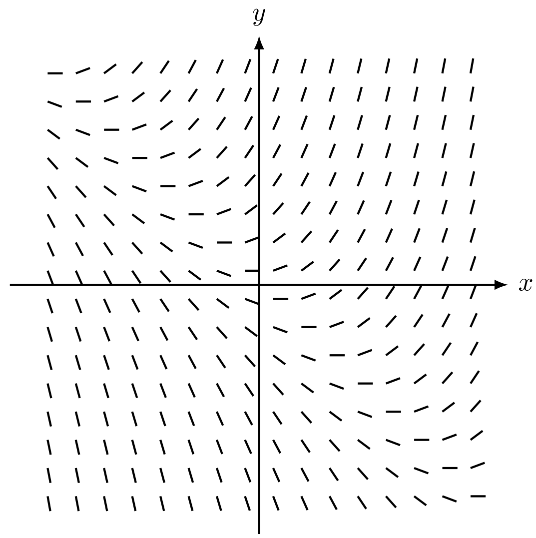

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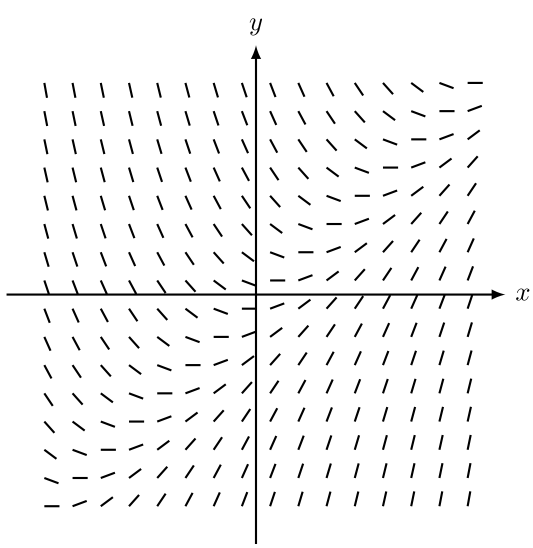

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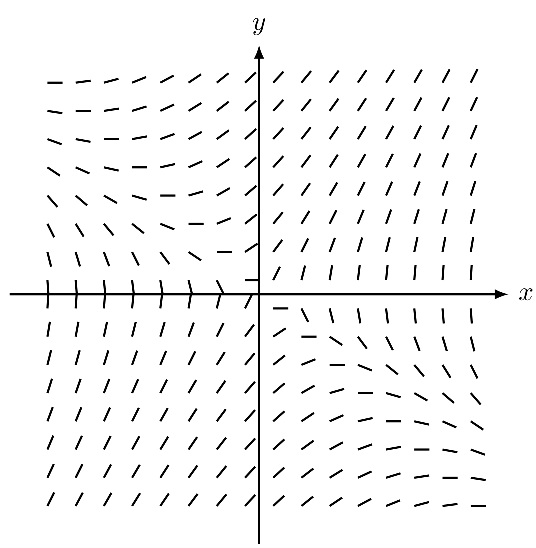


24. (★☆☆☆, 2 marks) Let $f(x, y) = \sin\left(x^2 - 6y\right)$. Calculate the partial derivative $f_y(0, \pi)$.

Answer:

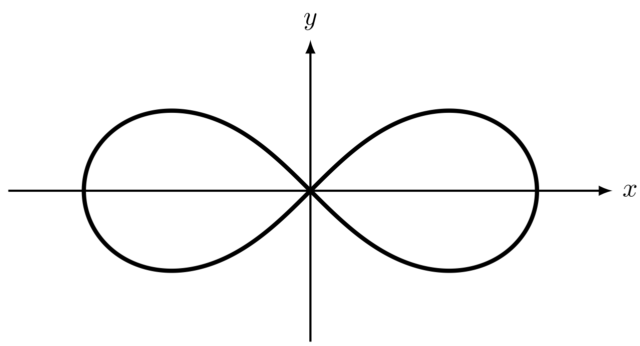
25. (★☆☆☆, 2 marks) Let $f(x,y) = 2x - 2x^2 - xy^2$. Find all the critical points. Your answer(s) should include both x- and y- values, and written in the form $(\_\_,\_\_)$.

Answer:

26. (★★☆☆, 10 marks) Find the equations of all tangent lines to the curve

$$\left(x^2 + y^2\right)^2 = 2\left(x^2 - y^2\right)$$

that are parallel to the line $y + 1 = 0$. All equations must be in the form $y = mx + b$. A picture of the curve is given below.



Answer:

27. (★★☆☆, 10 marks) Let

$$f(x) = 8xe^{-x/2}, \qquad x \geq 0.$$

(This is a scaled example of the *gamma distribution* from probability.) In this question, you may use without proof the fact that

$$f'(x) = -4e^{-x/2}(x-2).$$

(a) List the vertical and horizontal asymptotes. Leave the field blank if there is no such asymptote.

Note: For part (a), you may simply list your answers without justification.

Vertical asymptote(s):

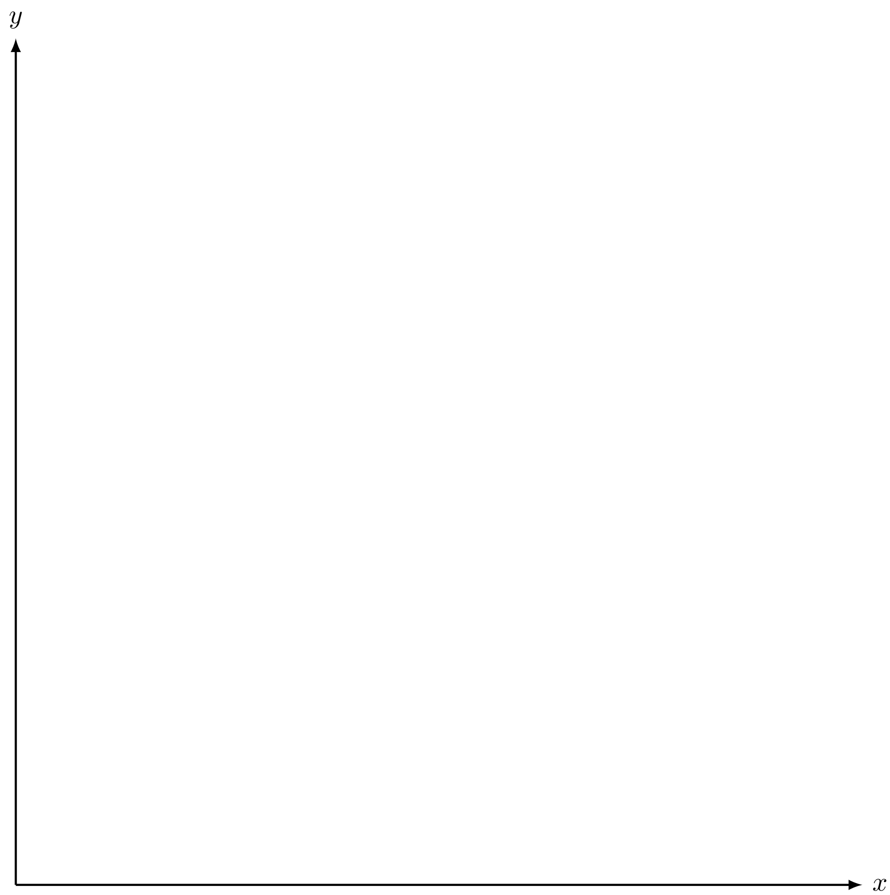
Horizontal asymptote(s):

(b) Determine the intervals where $f(x)$ is increasing and the intervals where $f(x)$ is decreasing. Leave the field blank if there is no such interval.

Increasing:

Decreasing:

(c) Sketch the graph of $f(x)$ on the axes below. Label all extrema and inflection points. Your label(s) should be at the point(s) in question, and coordinates should be written in the form (___, ___).



28. (★★☆☆, 10 marks) Doctor Roberts considers starting up a new factory making umbrellas. The factory would have fixed costs of \$5,000 per month and each umbrella would cost \$8 to produce. Market research indicates that if x thousand umbrellas are produced every month, they can be sold at price $p(x) = 30 - x$ dollars. Determine the optimal profit per month and the production level at which it occurs. Give units for your answers.

Maximum profit:

Production Level:

The next page is available if you need more space for your solution.

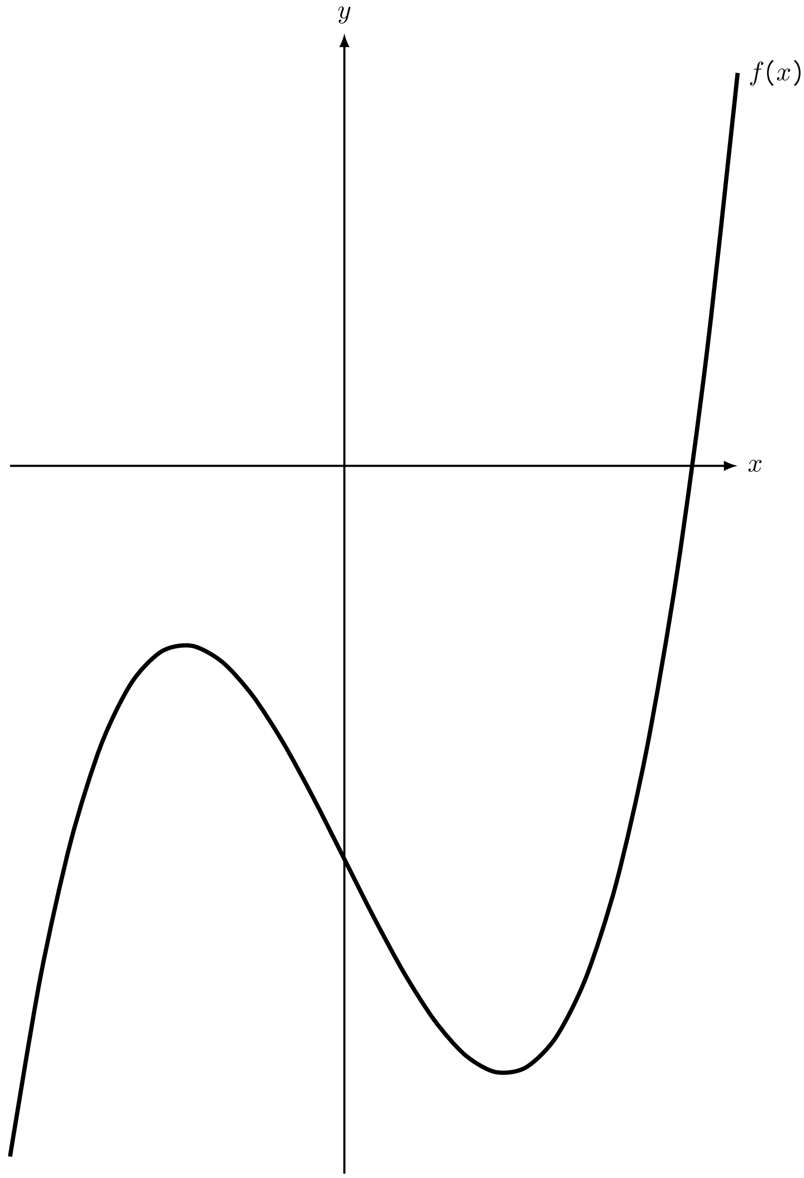
Additional page for your #28 solution if needed.

29. (★★★☆, 10 marks) Let $f(x) = x^3 - 2x - 2$.

(a) Let $x_0 = 0$. Use three iterations of Newton's Method to find x_1, x_2, and x_3.

Answer:

(b) Label x_0, x_1, x_2 and x_3 on the graph below. Will the iterations of Newton's method started in part (a) eventually converge toward the root? Justify your answer in 3-5 sentences.



30. (★★★★, 10 marks) Let $f(x)$ be a function with $f'(x), f''(x) > 0$ for all x. What is the maximum number of horizontal asymptotes that such a function $f(x)$ might have? What is the minimum number? Justify your answers carefully.

Maximum:

Minimum:

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