MATH 100 Final Exam
Duration: 2.5 hours

Student number								
Section								
Name	....							
Signature								

This test contains 30 questions on 38 pages worth a total of 100 marks.
Please ensure that your test is complete.

Rules regarding paper handling

1. Do not remove pages, staples, or otherwise disassemble your test.
2. You may request scrap paper. An invigilator will bring it to you. If you do not wish for this page to be graded mark it as "scrap."
3. You must submit all scrap paper whether you intend for it to be graded or not. No paper may be removed from the exam room.
4. This exam contains lots of blank space for your work. If you require additional paper, an invigilator will bring it to you. If you would like work on this additional paper to be graded you must include on each **side** of each extra page: (a) your unique 4 digit test number (b) the question number (c) your name (d) your student number. In particular, your **test number**, (a), can be found at the top centre of this page. Failure to include all of (a)-(d) will result in your extra page not being graded. Do not include work for more than one question on a single side of an extra sheet.

Rules governing UBC examinations:

1. Each candidate must be prepared to produce, upon request, a UBC card for identification.
2. No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination.
3. Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action:
   (a) Having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners;
   (b) Speaking or communicating with other candidates;
   (c) Purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received.
4. Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.
5. Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

Additional rules governing this examination:

1. This is a closed-book exam.
   (a) Calculators and other calculating devices may not be used.
   (b) Notes may not be used.
   (c) Watches must be removed and taken off the table.
2. If an answer box is provided, you must write down your answer (but not its justification) in the box.
3. Answers must be simplified and calculator-ready. For example, write $\log\left(e^{\sqrt{2}}\right) = \sqrt{2}$, but do not write $\sqrt{2} \approx 1.414$.
4. You must justify your answers unless an explicit exception is made.
5. You may use any result proven in class or on assignments.

1. (★☆☆☆, 2 marks) List the vertical and horizontal asymptotes of $y = \dfrac{x - 4}{\sqrt{4x^2 + 3}}$. Leave the field blank if there is no such asymptote.

Vertical asymptote(s):

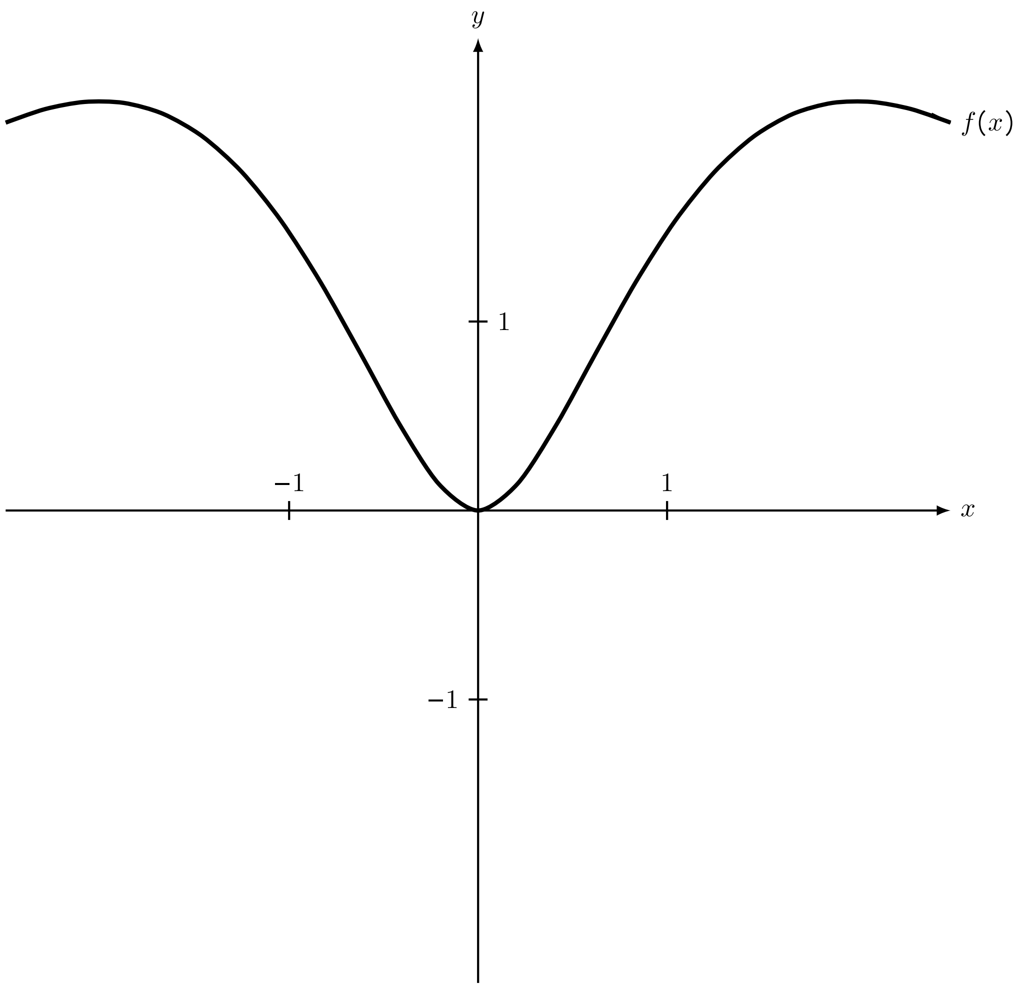
Horizontal asymptote(s):

2. (★☆☆☆, 2 marks) Find the value of the constant c that makes $f(x)$ continuous, where

$$f(x) = \begin{cases} cx + 3 & \text{if } x \leq 3 \\ cx^2 - 3 & \text{if } x > 3 \end{cases}.$$

Answer:

3. (★☆☆☆, 2 marks) A function $f(x)$ is pictured below. Sketch the graph of the derivative directly on the axes.



4. (★☆☆☆, 2 marks) Let $f(x) = -4e^{2x} + e^{-1}$. Find $f'(0)$.

Answer:

5. (★☆☆☆, 2 marks) Let $f(x) = x^x$. Find $f'(e)$.

Answer:

6. (★☆☆☆, 2 marks) Let $f(x) = \sqrt{4 + 5x^4}$. Find $f'(-1)$.

Answer:

7. (★☆☆☆, 2 marks) Let

$$x^2 (f(x))^2 = 9$$

and $f(3) = -1$. Find $f'(3)$.

Answer:

8. (★☆☆☆, 2 marks) Rewrite $f(x) = \cos(\arctan(4x))$ as a rational function, without any trigonometric functions.

Answer:

9. (★☆☆☆, 2 marks) Find the equation of the tangent line to the curve $y = x\sqrt{x}$ at the point $(1, 1)$. Your answer must be written in the form $y = mx + b$.

Answer:

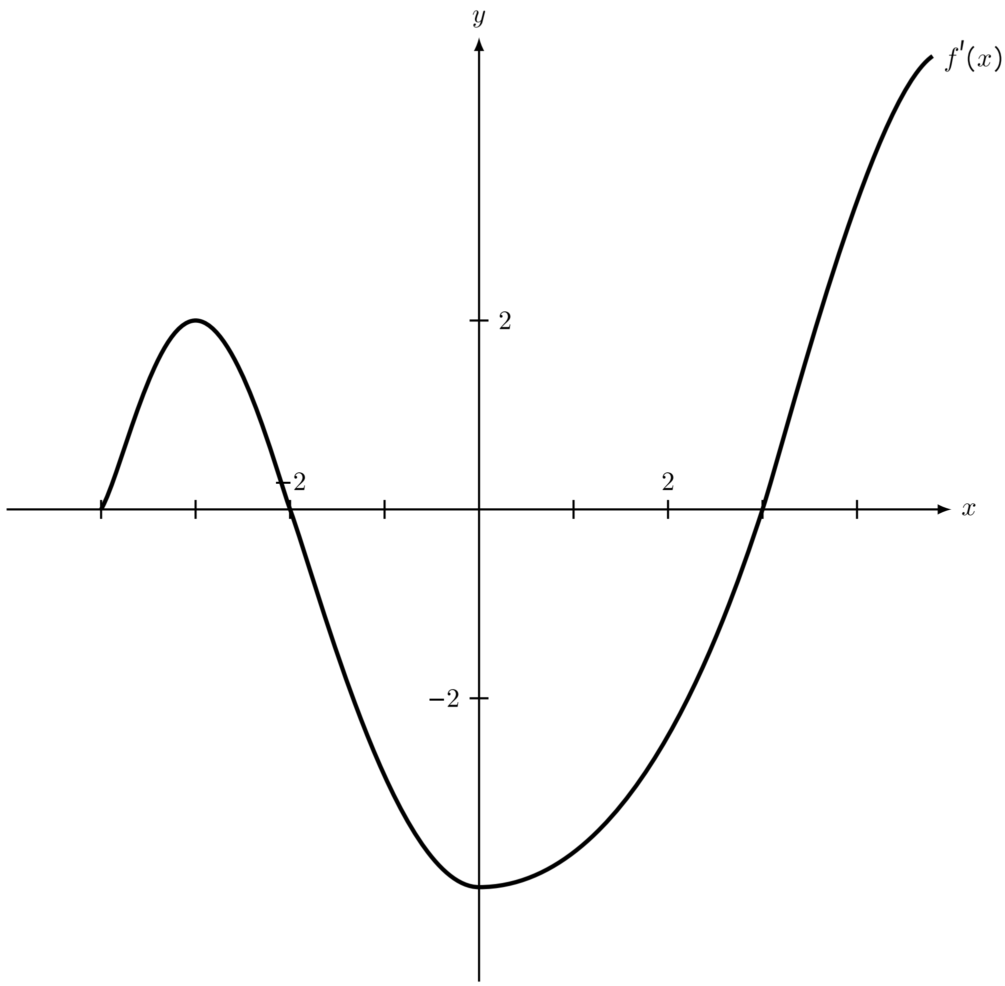
10. (★☆☆☆, 2 marks) Let A be the area of a circle with radius r. If r changes at a rate of $2\,\mathrm{cm/s}$, at what rate is the area enclosed by the circle changing when $r = 3\,\mathrm{cm}$?

Answer:

11. (★☆☆☆, 2 marks) A car is driving at 20 m/s along a straight road. A police officer is standing 4 m to the side of the road with a radar gun measuring speeds. The measured speed is the rate of change of the distance between car and officer. What is the measured speed when the car has not yet passed the officer and is 3 m away from the closest point on the road to the officer?

Answer:

12. (★☆☆☆, 2 marks) Consider the following graph of a *derivative* $f'(x)$.



Find all the intervals where $f(x)$ is concave down.

Answer:

13. (★☆☆☆, 2 marks) Let $f(x) = x^4 - 4x^3 - 18x^2 + 13$. Find all intervals where $f(x)$ is concave up.

Answer:

14. (★☆☆☆, 2 marks) Let

$$f(x) = x + \frac{4}{x}$$

be defined on the domain $x > 0$. Find the x-values of all extrema on that interval, indicating for each x-value found whether there is a local maximum or a local minimum there.

Answer:

15. (★☆☆☆, 2 marks) Use the linear approximation of $f(x) = \log(x)$ at $x = 1$ to estimate $\log(1.06)$.

Answer:

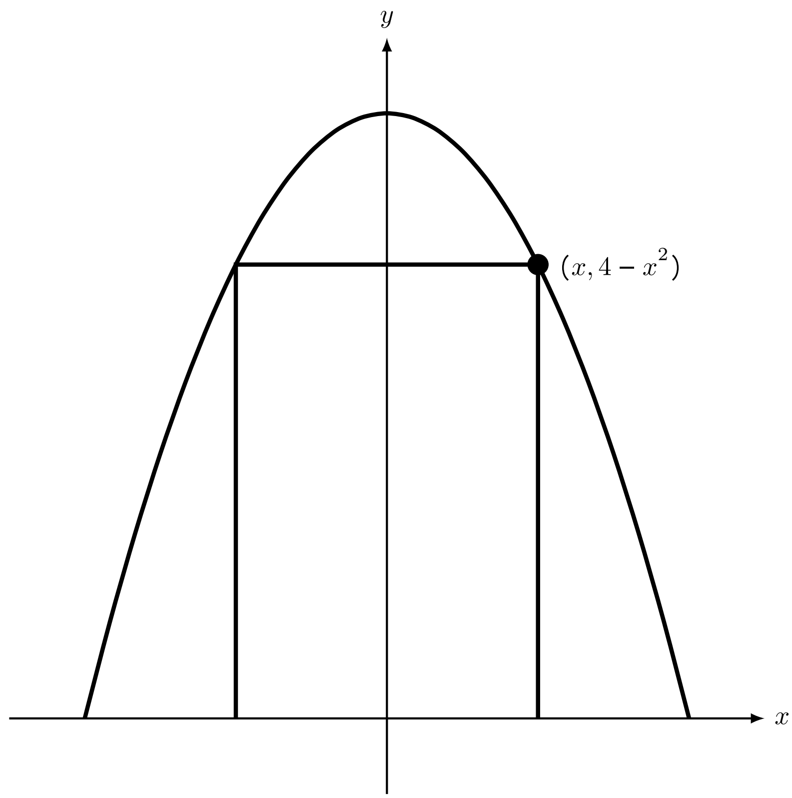
16. (★☆☆☆, 2 marks) Find the coefficient of the x^3 term of the Taylor polynomial of $f(x) = \dfrac{e^x}{1-x}$ about the point $x = 0$.

Answer:

17. (★☆☆☆, 2 marks) At what value of x on the curve $y = x + x^2 - x^3$ does the tangent line have the largest positive slope?

Answer:

18. (★☆☆☆, 2 marks) A rectangle is inscribed with its base on the x-axis and its upper corners on the parabola $y = 4 - x^2$, as shown below.



What is the height of such a rectangle with the greatest possible area?

Answer:

19. (★☆☆☆, 2 marks) Find all values of r such that $y = e^{rt}$ solves the differential equation

$$y'' + 6y' + 5y = 0.$$

Answer:

20. (★☆☆☆, 2 marks) Sketch the phase line for the differential equation

$$y' = -y(y-2)(y-10).$$

Clearly indicate the steady states (also called equilibrium points), and indicate with arrows whether y is increasing or decreasing on either side of each steady state.

y

21. (★☆☆☆, 2 marks) Estimate the solution to the differential equation

$$y' = -y(y-2)(y-10)$$

using one step of size $\Delta t = 1/10$ with an initial condition $y(0) = -1$.

Answer:

22. (★☆☆☆, 2 marks) Let $f(x, y) = \sin\left(x^2 - 6y\right)$. Calculate the partial derivative $f_y(0, \pi)$.

Answer:

23. (★☆☆☆, 2 marks) Let $f(x, y) = 2x - 2x^2 - xy^2$. Find all the critical points. Your answer(s) should include both x- and y- values, and written in the form $(\_\_, \_\_)$.

Answer:

24. (★☆☆☆, 2 marks) Find the absolute maximum of the function

$$f(x, y) = 2x - 3y$$

on the rectangular domain $0 \le x \le 2$ and $0 \le y \le 1$.

Answer:

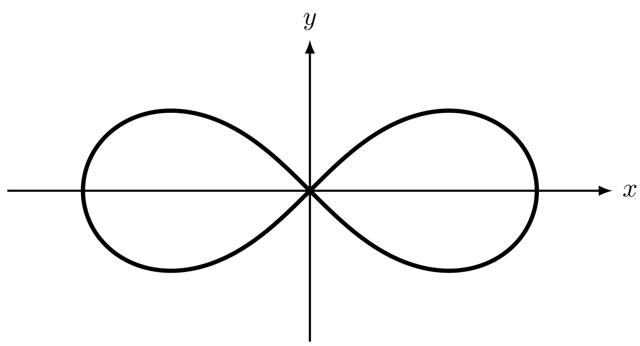
25. (★☆☆☆, 2 marks) Find the value of a that minimizes the sum of squared residuals for the model $y = ax$ fitting the data points (2,2), (3,2).

Answer:

26. (★★☆☆, 10 marks) Find the equations of all tangent lines to the curve

$$\left(x^2 + y^2\right)^2 = 2\left(x^2 - y^2\right)$$

that are parallel to the line $y + 1 = 0$. All equations must be in the form $y = mx + b$. A picture of the curve is given below.



Answer:

27. (★★☆☆, 10 marks) Let

$$f(x) = 8xe^{-x/2}, \qquad x \geq 0.$$

(This is a scaled example of the *gamma distribution* from probability.) In this question, you may use without proof the fact that

$$f'(x) = -4e^{-x/2}(x-2).$$

(a) List the vertical and horizontal asymptotes. Leave the field blank if there is no such asymptote.

Note: For part (a), you may simply list your answers without justification.

Vertical asymptote(s):

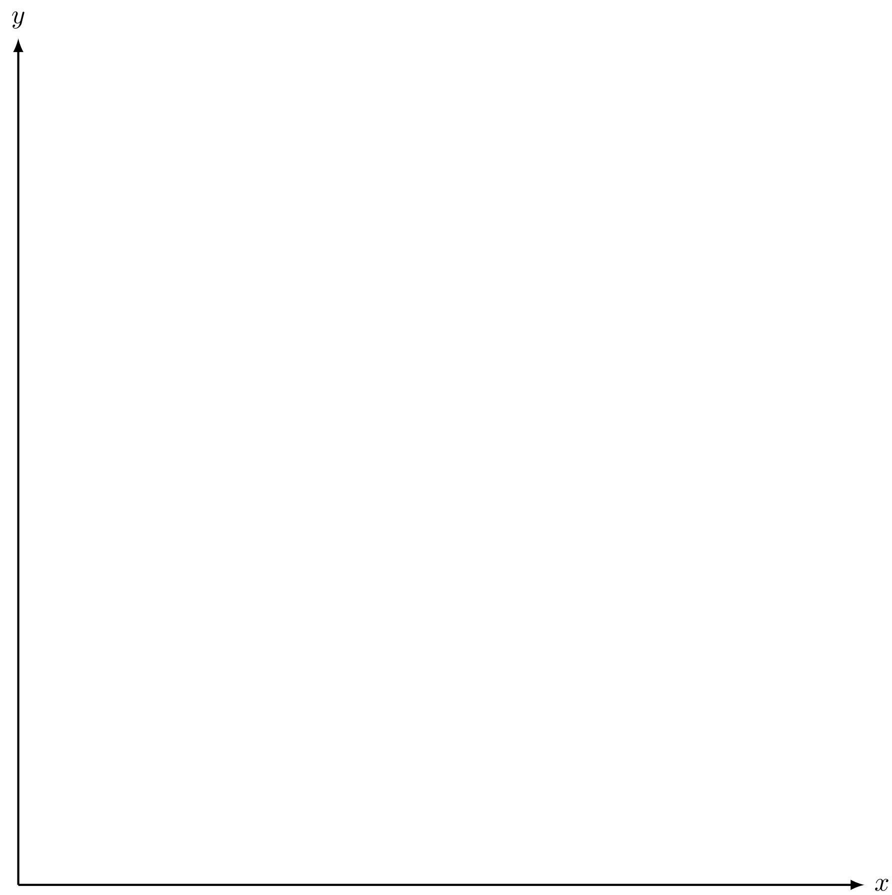
Horizontal asymptote(s):

(b) Determine the intervals where $f(x)$ is increasing and the intervals where $f(x)$ is decreasing. Leave the field blank if there is no such interval.

Increasing:

Decreasing:

(c) Sketch the graph of $f(x)$ on the axes below. Label all extrema and inflection points. Your label(s) should be at the point(s) in question, and coordinates should be written in the form $(\_\_, \_\_)$.



28. (★★★☆, 10 marks) Let $f(x) = x^3 - 2x - 2$.

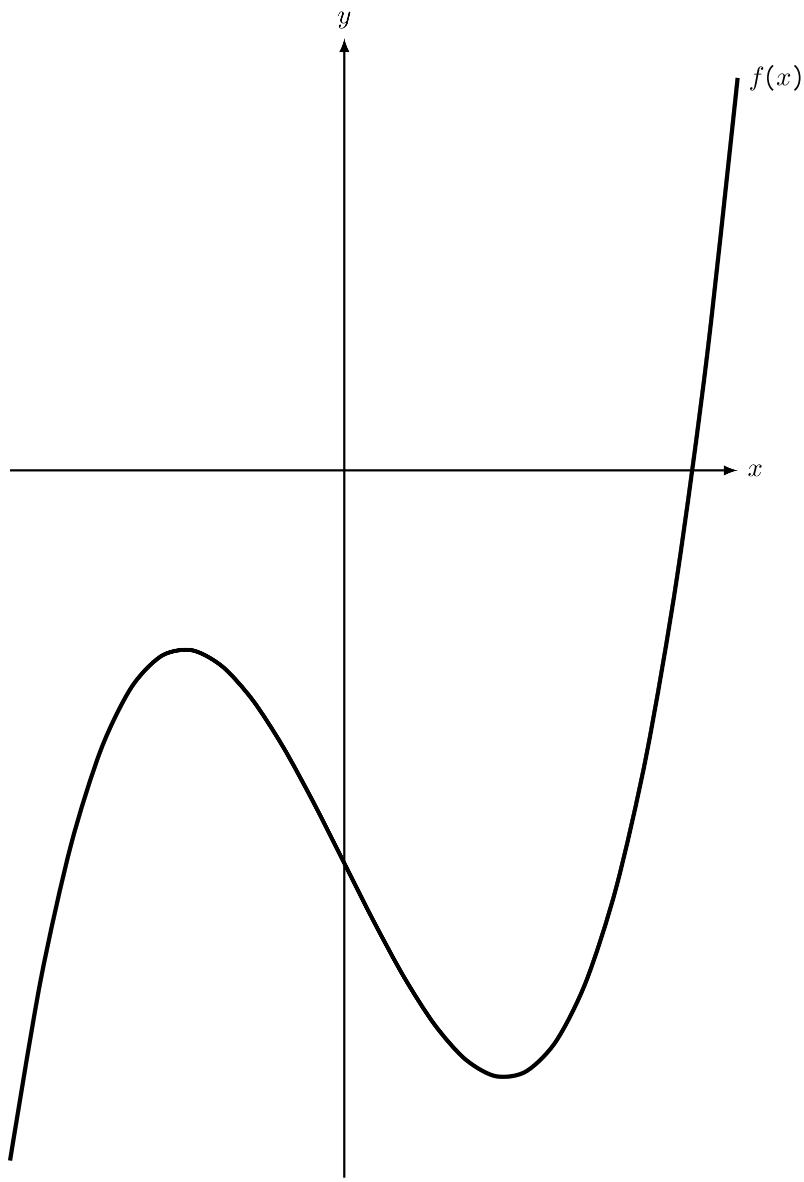
(a) Let $x_0 = 0$. Use three iterations of Newton's Method to find x_1, x_2, and x_3.

$x_1 =$

$x_2 =$

$x_3 =$

(b) Label x_0, x_1, x_2 and x_3 on the graph below. Will the iterations of Newton's method started in part (a) eventually converge toward the root? Justify your answer in no more than 3 sentences.



29. (★★★☆, 10 marks) Consider the differential equation from questions 20 and 21:

$$y' = -y(y-2)(y-10).$$

Redraw your phase line from question 20 on the line below so that you can get part marks here if your answer to that question is incorrect.

y

(a) Sketch the solution to the differential equation for each of the initial conditions $y(0) = -1$, $y(0) = 3$ on the same graph. Any inflection points should be visible but can be positioned roughly.

(b) Comparing your sketch for the initial condition $y(0) = -1$ against your calculation in question 21, you should find them to be inconsistent. Explain in what way they are inconsistent and what you might do differently to fix that.

30. (★★☆☆ excluding part (f), 10 marks) A drug used to treat cancer is effective at low doses with an efficacy (ability to cure) that increases with the quantity of the drug used for treatment. However, at sufficiently high doses, the drug becomes toxic (can cause death). The fraction of patients surviving cancer with this drug treatment is given by

$$S(x) = \frac{x}{k_1 + x} - \frac{x}{k_2 + x}$$

where x is the drug quantity in hundreds of milligrams/day given to the patient and $0 < k_1 < k_2$.

(a) Find $S(0)$. **Explain what your finding means** in medical terms.

$S(0) =$

(b) Find $\lim_{x\to\infty} S(x)$. **Explain what your finding means** in medical terms .

$\lim_{x\to\infty} S(x)=$

(c) Find $R(x) = \lim_{k_2\to\infty} S(x)$.

$R(x)=$

(d) Find $Q(x) = \lim_{k_2\to k_1} S(x)$.

$Q(x)=$

(e) Interpret what k_2 represents in medical terms. What do high and low values of k_2 correspond to?

(f) (★★★★) For this part, let $k_1 = 1$ and, for notational simplicity, rename $k_2 = k$. What is the optimal daily drug quantity to administer so as to maximize the fraction of patients surviving? Justify **carefully** that you have indeed found a global maximum.

*This blank page is for your solution to **Question 30**, if you need more space.*