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Mathematics 100 and 180 Final Exam
Duration 150 minutes

Monday December 12$^{\text{th}}$ 2016

- Do not circle the boxes. Use **dark pen/pencils** to indicate your choice.

excellent

wrong, lines too thin

too light

really bad

- Do not write or mark in the shaded areas labelled *'For marker use only'*.
- Do not write or mark in the area around the dots in the corners of each page.

Please encode your student number below	Please write your details below
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There are 14 questions worth a total of 100 marks.

Please read the instructions on the next page before you start.

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Please read before you start

- Read all the questions carefully before starting to work.
- Q1–Q8 are multiple choice questions; shade in the appropriate box or boxes.
  — we recommend that you use a dark pencil for these questions.
- Q9–Q14 and later are long-answer; you should give complete arguments and explanations for all your calculations; answers without justifications will not be marked.
- Q13(b) and Q14(b) are challenging questions — we recommend that you attempt them last.
- This is a closed-book examination. **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)
- **If you need more space:**
  - There are blank pages at the end of the test.
  - At the original question you must indicate to the marker that you continued at one of these blank pages.

Student Conduct during Examinations

1. Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
2. Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
3. No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no examination candidate shall be permitted to enter the examination room once the examination has begun.
4. Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
5. Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
   (i) speaking or communicating with other examination candidates, unless otherwise authorized;
   (ii) purposely exposing written papers to the view of other examination candidates or imaging devices;
   (iii) purposely viewing the written papers of other examination candidates;
   (iv) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
   (v) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
6. Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
7. Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
8. Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

1. **Multiple choice questions — each part is worth 2 marks**
   Shade exactly one box in each part.

Q(1.1) : Evaluate $\lim_{x \to +\infty} \dfrac{3x+1}{\sqrt{4x^2 - 3x - 7}}$.

- ☐ ∞
- ☐ $-\dfrac{3}{2}$
- ☐ $-\dfrac{3}{4}$
- ☐ 1
- ☐ $\dfrac{3}{4}$
- ☐ $\dfrac{3}{2}$

Q(1.2) : Evaluate $\lim_{x \to 0^-} \dfrac{|x|}{x}$.

- ☐ -1
- ☐ 1
- ☐ $-\infty$
- ☐ ∞
- ☐ Cannot be determined
- ☐ 0

Q(1.3) : Where is $f(x) = \dfrac{\sin\left(\frac{\pi x}{2}\right)}{\sqrt{1-x^2}}$ continuous?

- ☐ $(-\pi/2, \pi/2)$
- ☐ $(-\infty, -1) \cup (1, \infty)$
- ☐ Everywhere except $x = \pm 1$
- ☐ $[-1, 1]$
- ☐ $[-\pi/2, \pi/2]$
- ☐ $(-1, 1)$

2. **Multiple choice questions — each part is worth 2 marks**
   Shade exactly one box in each part.

Q(2.1) : Find the derivative of $f(x) = x^2e^x$.

- ☐ $xe^x(x+2)$
- ☐ $2xe^{x-1} + x^2e^x$
- ☐ $e^x(2x+1)$
- ☐ $e^x(x+2)$
- ☐ $2xe^x + x^2e^{x-1}$
- ☐ $xe^x(2x+1)$

Q(2.2) : Find the derivative of $g(x) = \dfrac{x^2+3}{2x-1}$.

- ☐ $\dfrac{2(x^2+x-3)}{(2x-1)^2}$
- ☐ $\dfrac{2(x^2+x+3)}{(2x-1)^2}$
- ☐ $\dfrac{2(x^2-x-3)}{(2x-1)^2}$
- ☐ $\dfrac{(x^2+x+3)}{(2x-1)^2}$
- ☐ $\dfrac{2(x^2-x+3)}{(2x-1)^2}$
- ☐ $\dfrac{(x^2-x-3)}{(2x-1)^2}$

Q(2.3) : Find the derivative of $h(x) = \log(\sin(x))$. Remember that $\log x = \log_e x = \ln x$.

- ☐ $\log(\cos(x))$
- ☐ $\dfrac{1}{x\sin(x)}$
- ☐ $\dfrac{1}{\sin(x)\cos(x)}$
- ☐ $\dfrac{\cos(x)}{x}$
- ☐ $\dfrac{1}{\sin(x)}$
- ☐ $\dfrac{\cos(x)}{\sin(x)}$

3. **Multiple choice questions — each part is worth 2 marks**
   Shade exactly one box in each part.

Q(3.1) : Compute the derivative of $f(x) = x^{x-1}$. Remember that $\log x = \log_e x = \ln x$.

- ☐ $x^{x-1}\left(\log(x) + \dfrac{x}{x-1}\right)$
- ☐ $x^{x-1}\left(\log(x-1) + \dfrac{x-1}{x}\right)$
- ☐ $x^{x-1}\left(\log(x) + \dfrac{x-1}{x}\right)$
- ☐ $x^{x-1}\left(\log(x-1) + \dfrac{x}{x-1}\right)$
- ☐ $x^{x-1}\left(x(x-1) + \dfrac{x}{x-1}\right)$
- ☐ $x^{x-1}\left(x(x-1) + \dfrac{x-1}{x}\right)$

Q(3.2) : A scientist isolates 32 grams of a radioactive substance in the lab at 1PM. At 5PM it weighs 4 grams. What is the half-life of the substance?

- ☐ 45 minutes
- ☐ 100 minutes
- ☐ 90 minutes
- ☐ 120 minutes
- ☐ 80 minutes
- ☐ 60 minutes

Q(3.3) : Approximate $(26)^{1/3}$ using a linear approximation of the function $h(x) = x^{1/3}$.

- ☐ 79/27
- ☐ 26/9
- ☐ 80/27
- ☐ 83/27
- ☐ 82/27
- ☐ 28/9

Space for your work — NOTHING written on this page will be marked.

4. **Multiple choice questions — each part is worth 2 marks**
Shade exactly one box in each part.

Q(4.1) : Consider the line $y = 4x + 2$. To which of the following functions is it tangent at $x = 1$?

- ☐ $f(x) = x^3 + 2x^2 + 3x$
- ☐ $f(x) = x^3 + x^2 - x$
- ☐ $f(x) = x^2 + 3x + 2$
- ☐ $f(x) = x^3 + x + 2$
- ☐ $f(x) = 2\sqrt{x+3} + 2$
- ☐ None of these

Q(4.2) : Simplify $\sin(\arctan(4))$.

- ☐ $\dfrac{1}{\sqrt{17}}$
- ☐ $\dfrac{4}{\sqrt{5}}$
- ☐ $\dfrac{1}{\sqrt{5}}$
- ☐ $\dfrac{4}{\sqrt{17}}$
- ☐ $\dfrac{1}{4}$
- ☐ 4

Q(4.3) : Let $f(x)$ be a continuous function defined for all real numbers x. Suppose $f(x)$ is increasing on the intervals $(-\infty, -1)$ and $(3, \infty)$, decreasing on $(-1, 3)$, $f(-1) = 2$ and $f(3) = 1$. How many zeroes does $f(x)$ have?

- ☐ 1
- ☐ 3
- ☐ 2
- ☐ Cannot determine from the information given.
- ☐ 0

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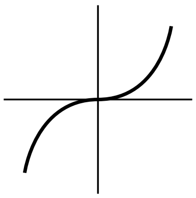
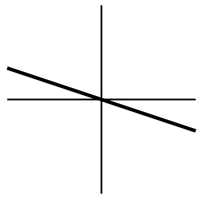
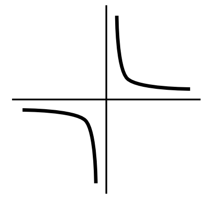
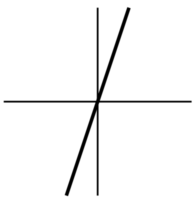
5. **Multiple choice questions — each part is worth 2 marks**
Shade exactly one box in each part.

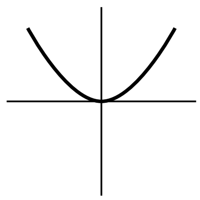
Q(5.1) : Which of the following graphs is a good approximation of $f(x) = \dfrac{x^2 - 2x}{x + 7}$ for x on the small interval $[-1/10, 1/10]$?

☐ 

☐ 

☐ 

☐ 

☐ 

☐ None of these

Q(5.2) : Let x, y satisfy the equation $x^y = 5y - x + 1$. Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ at the point $(x, y) = (1, 1/5)$.

☐ $1/25$	☐ $6/25$	☐ $25/6$
☐ 25	☐ $6/5$	☐ $5/6$

Q(5.3) : Compute the limit $\displaystyle\lim_{x\to 0} \frac{1 - \cos^2 x}{x^2}$.

☐ 1	☐ $1/2$	☐ 0
☐ $-1/2$	☐ ∞	☐ -1

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6. **Multiple choice questions — each part is worth 2 marks**
Shade exactly one box in each part.

Q(6.1) : Which of the following is the most general antiderivative of the function e^{2x+3}? In the functions below, c is an arbitrary constant.

- ☐ $\frac{1}{3}e^{2x+3} + c$
- ☐ $\frac{1}{2}e^{2x+3} + c$
- ☐ $e^{2x+3} + c$
- ☐ $3e^{2x+3} + c$
- ☐ $(2x+3)e^{2x+2} + c$
- ☐ $2e^{2x+3} + c$

Q(6.2) : An object is thrown straight up in the air at $t = 0$ seconds. Its height in metres at t seconds is given by

$$h(t) = s_0 + v_0 t - 5t^2$$

where s_0 and v_0 are constants. In the first second the object rises 5 metres. For how many seconds does the object rise before starting to fall back down?

- ☐ 1 second
- ☐ 2 seconds
- ☐ 3 seconds
- ☐ 15 seconds
- ☐ 10 seconds
- ☐ 5 seconds

7. **Multiple choice questions. Each part is worth 4 marks if there are no errors, 2 marks if there is 1 error, and 0 marks otherwise.**

Q(7.1) : Let $f(x)$ be a continuous function on the open interval (a, b). Which of the following four statements are always true?
Select all that apply.

(A) If $\lim_{x \to a+} f(x) = -\infty$ and $\lim_{x \to b-} f(x) = \infty$, then there is at least one zero of $f(x)$ inside (a, b).

(B) If $f(x)$ has a local minimum at c in (a, b), then $f'(c) = 0$.

(C) $f(x)$ must have both maximum and minimum inside (a, b).

(D) If $f''(c) > 0$ for some point c in (a, b), then $f(x)$ has a local minimum at c.

- ☐ A
- ☐ B
- ☐ C
- ☐ D
- ☐ None of these answers are correct.

Q(7.2) : Which of the following five functions are concave up on their whole domain?
Select all that apply.

(A) $f(x) = \cos(x) + x^2$

(B) $f(x) = e^{-x^2}$

(C) $f(x) = x^4 + 2x^2 - 5x + 1$

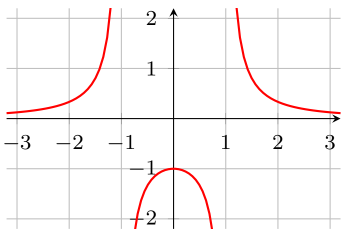
(D) $f(x) = 1 - x^2$

(E) $f(x) = -\log(x)$ Recall $\log x = \log_e x = \ln x$.

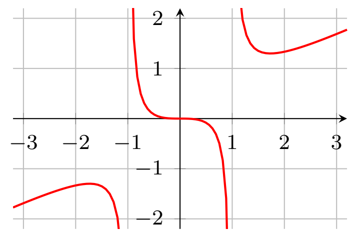
- ☐ A
- ☐ B
- ☐ C
- ☐ D
- ☐ E
- ☐ None of these answers are correct.

8. **Multiple choice question — 5 marks:** Consider the following six graphs:

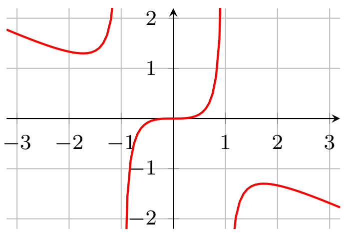
A



B



C



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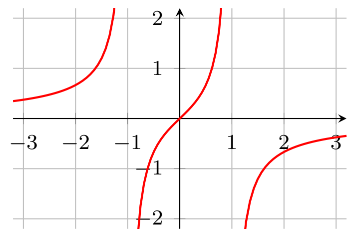

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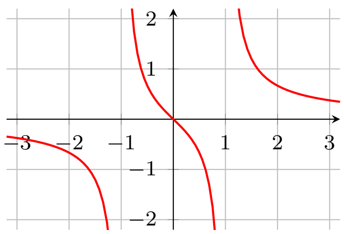

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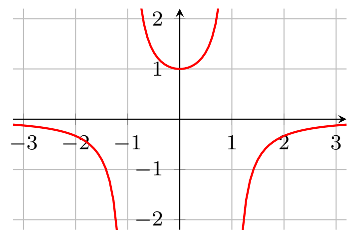

Match the following five functions with the above graphs:

$\dfrac{1}{1-x^2}$: ☐A ☐B ☐C ☐D ☐E ☐F

$\dfrac{x}{x^2-1}$: ☐A ☐B ☐C ☐D ☐E ☐F

$\dfrac{x^3}{2(1-x^2)}$: ☐A ☐B ☐C ☐D ☐E ☐F

$\dfrac{1}{x^2-1}$: ☐A ☐B ☐C ☐D ☐E ☐F

$\dfrac{x^3}{2(x^2-1)}$: ☐A ☐B ☐C ☐D ☐E ☐F

0 1 2 3 4 5 — **For marker use only**

Q9(a) Let $f(x) = \dfrac{\sqrt{x^2 - 8}}{x - 4}$.

2 marks: What is the domain of $f(x)$?

1 mark: Give all intercepts of $f(x)$.

2 marks: What are the horizontal asymptotes of $f(x)$, if any?

$\square$ 0 $\square$ 1 $\square$ 2 $\square$ 3 $\square$ 4 — **For marker use only**

Q9(b) Let $f(x) = \dfrac{\sqrt{x^2 - 8}}{x - 4}$.

1 mark: Compute and simplify $f'(x)$.

1 mark: Find all intervals where $f(x)$ is increasing.

2 marks: Find all intervals where $f(x)$ is decreasing.

□0 □1 □2 □3 □4 — **For marker use only**

Q9(c) Let $f(x) = \dfrac{\sqrt{x^2-8}}{x-4}$. Note that

$$f''(x) = \frac{8x^2(x-3)}{(x^2-8)^{3/2}(x-4)^3}.$$

1 mark: Find all intervals where $f(x)$ is concave down.

2 marks: Find all intervals where $f(x)$ is concave up.

1 mark: Find the x-coordinates of all inflection points of $f(x)$.

Q10(a) — 4 marks: Determine whether the following function is continuous at $x = 0$ using the definition of continuity. You must fully justify your solution.

$$g(x) = \begin{cases} \dfrac{\sqrt{x^2+1}-1}{x}, & x < 0 \\ 0, & x \geq 0 \end{cases}$$

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Q10(b) — 4 marks: Let $f(x) = \dfrac{x}{x-2}$. Compute $\dfrac{\mathrm{d}f}{\mathrm{d}x}$ using the definition of the derivative. **No marks will be given for the use of derivative rules,** but you may use them to check your answer.

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Q11: A and B, two people of identical height h, stand beneath a street lamp of height L. A walks in a straight line and at a constant speed away from the street lamp. One second later, B walks in a straight line and at the same speed, but in the opposite direction, away from the street lamp. As A and B move away from the lamp, their shadows grow longer.

2 marks: Let a be the length of A's shadow, and b be the length of B's shadow. Let x be the distance A has walked, and y be the distance B has walked. Draw and label a picture that illustrates the scenario two seconds after A begins to walk away from the street lamp. Your picture should indicate all relevant lengths and the associated variables.

4 marks: As A and B walk away from the lamp, both their shadows are getting longer. Whose shadow is changing length faster, two seconds after A left the lamp? Justify your answer.

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This page is for Question 11 only.
Work for Q11 on this page will be graded.

Q12 — 6 marks: Consider a cone with height 2 metres and whose circular base has radius 1 metre. Find the dimensions of the circular cylinder of largest volume that can be contained in the cone. (The base of the cylinder lies at the base of the cone.)

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This page is for Question 12 only.
Work for Q12 on this page will be graded.

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Q13(a) — 4 marks: Let $T_3(x)$ be the third degree Taylor polynomial centred at $x = 0$ for

$$f(x) = (x + 1)\sin(x).$$

Write down $T_3(x)$. Make sure that you simplify the coefficients.

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Q13(b) — 6 marks — challenging. Now let $T_n(x)$ be the nth degree Taylor polynomial centred at $x = 1$ for the function

$$f(x) = \log(x). \qquad \text{Remember that } \log x = \log_e x = \ln x.$$

For which value(s) of n will $T_n(1.1)$ give an **underestimate** of $\log(1.1)$? You must justify your answer.

☐0 ☐1 ☐2 ☐3 ☐4 — **For marker use only**

Q14(a) — 4 marks: Let $f(x)$ be a function so that

- $f(x), f'(x),\ f''(x)$ exist and are continuous for all x, and
- $|f(x) - \sin x| \leq 1/3$ for all x.

Show that $f(x)$ has at least one zero in the open interval $(0, 2\pi)$.

$\square$0 $\square$1 $\square$2 $\square$3 $\square$4 $\square$5 $\square$6 — **For marker use only**

Q14(b) — 6 marks — challenging. Let $f(x)$ be a function so that

- $f(x), f'(x), f''(x)$ exist and are continuous for all x, and
- $|f(x) - \sin x| \leq 1/3$ for all x.

Show that $f''(x)$ has at least one zero in the open interval $(0, 2\pi)$.

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