The University of British Columbia Department of Mathematics

Qualifying Examination—Linear and Abstract Algebra

January 13, 2024

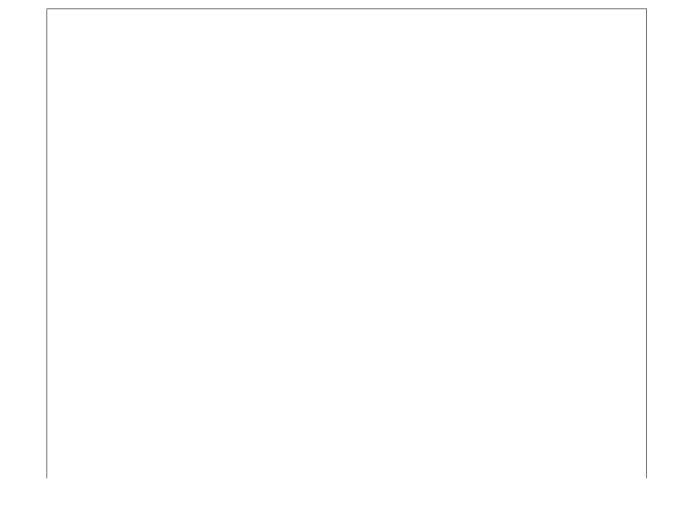
Linear Algebra

1. (10 points) Consider the linear system

$$Ax = b$$
 with $A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

- (a) [5 points] For what vectors b will this system have at least one solution? Find an orthogonal basis for this set.
- (b) [5points] Suppose $b = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$. Is a solution possible? If so, characterize all possible solutions. If not

solution is possible, find the best approximation, i.e. an x that minimizes |Ax - b|. If this approximate solution is not unique, characterize all possible optimal solutions.



2. (10 points) Consider the map

$$D: f(t) \to ct \frac{df}{dt} + \frac{d^2f}{dt^2}$$

D defines a linear map of the space L_n of polynomials f(t) of degree $\leq n$ into itself, and $c \in \mathbb{R}$ is a parameter.

- (a) [3 points] What is the rank of D, depending on c?
- (b) [4 points] What are the eigenvalues of D, depending on c?
- (c) [3 points] Find three non-zero eigenvectors of D for $c \neq 0.$

3. (10 points) Recall that for a symmetric matrix A we write A>0 to mean that A is positive definite, i.e. $x^TAx>0$ for all vectors x. If A and B are $n\times n$ real symmetric matrices such that A-B is positive

 (a) [4 points] If A > 0, show that there exists a symmetric matrix L > 0 such that L² = A. (b) [4 points] If A, B > 0 and A > I, where I is the identity matrix, show that A⁻¹ < I. (c) [2 points] f A, B > 0 and A > B, show that A⁻¹ < B⁻¹. 					

Abstract Algebra

definite, then we write A > B.

In this section, you can use any theorem from group theory, commutative algebra, Galois theory, etc. without proof as long as you state it clearly.

4. (13 points) Let R be a commutative ring with 1, and let M be a finitely generated module over R. Recall that for a subset $S \subset M$, its annihilator is the set $\mathrm{Ann}(S) = \{r \in R : rx = 0 \ \forall x \in S\} \subset R$.

The parts of this problem can be solved independently; if you do not know how to solve one of them, still try the later ones.

- (a) Prove that if R = k[t], the ring of polynomials over a field k, Ann(M) is an ideal generated by some polynomial $f \in k[t]$. Restate this result as a statement about a linear operator acting on a vector space.
- (b) Let R = k[t] and let M be a finitely generated R-module, as above. Suppose that $f = \prod_{i=1}^{\ell} p_i^{a_i}$ is a generator of $\operatorname{Ann}(M)$, where p_i are irreducible elements of k[t]. Prove that M can be decomposed as a direct sum of submodules, $M = M_1 \oplus \cdots \oplus M_\ell$, such that $\operatorname{Ann}(M_i) = (p_i^{a_i})$.
- (c) State an analogous statement over $\mathbb Z$ instead of k[t]. Is it true?
- (d) State (carefully) an analogous statement for $R = \mathbb{Z}[\sqrt{-5}]$. Is it true?

5.	 (10 points) (a) Let p be a prime and let H = Z/pZ × · · · × Z/pZ be the product of n factors. Prove that the group Aut(H) of automorphisms of H is isomorphic to GL_n(F_p), where F_p is the field of p elements. (b) Find the order of GL₂(F_p). (Bonus: find the order of GL_n(F_p).)
	 (c) Assuming that GL₃(F₂) = 168, prove that there exists a group of order 56 with a non-normal subgroup of order 7.
6.	(7 points) Let $p(x) = x^3 - 4x + 2$.
	(a) Is this polynomial irreducible over Q?(b) Find the degree of its splitting field over Q.