The University of British Columbia
Department of Mathematics
Qualifying Examination—Linear and Abstract Algebra
January 13, 2024

Linear Algebra

1. (10 points) Consider the linear system

$$Ax = b \qquad \text{with } A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

(a) [5 points] For what vectors b will this system have at least one solution? Find an orthogonal basis for this set.

(b) [5points] Suppose $b = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$. Is a solution possible? If so, characterize all possible solutions. If not solution is possible, find the best approximation, i.e. an x that minimizes $|Ax - b|$. If this approximate solution is not unique, characterize all possible optimal solutions.

2. (10 points) Consider the map

$$D : f(t) \to ct\frac{df}{dt} + \frac{d^2 f}{dt^2}$$

D defines a linear map of the space L_n of polynomials $f(t)$ of degree $\leq n$ into itself, and $c \in \mathbb{R}$ is a parameter.

(a) [3 points] What is the rank of D, depending on c?

(b) [4 points] What are the eigenvalues of D, depending on c?

(c) [3 points] Find three non-zero eigenvectors of D for $c \neq 0$.

3. (10 points) Recall that for a symmetric matrix A we write $A > 0$ to mean that A is positive definite, i.e. $x^T Ax > 0$ for all vectors x. If A and B are $n \times n$ real symmetric matrices such that $A - B$ is positive

definite, then we write $A > B$.

(a) [4 points] If $A > 0$, show that there exists a symmetric matrix $L > 0$ such that $L^2 = A$.

(b) [4 points] If $A, B > 0$ and $A > I$, where I is the identity matrix, show that $A^{-1} < I$.

(c) [2 points] f $A, B > 0$ and $A > B$, show that $A^{-1} < B^{-1}$.

Abstract Algebra

In this section, you can use any theorem from group theory, commutative algebra, Galois theory, etc. without proof as long as you state it clearly.

4. (13 points) Let R be a commutative ring with 1, and let M be a finitely generated module over R. Recall that for a subset $S \subset M$, its annihilator is the set $\text{Ann}(S) = \{r \in R : rx = 0 \ \forall x \in S\} \subset R$.

The parts of this problem can be solved independently; if you do not know how to solve one of them, still try the later ones.

(a) Prove that if $R = k[t]$, the ring of polynomials over a field k, $\mathrm{Ann}(M)$ is an ideal generated by some polynomial $f \in k[t]$. Restate this result as a statement about a linear operator acting on a vector space.

(b) Let $R = k[t]$ and let M be a finitely generated R-module, as above. Suppose that $f = \prod_{i=1}^{\ell} p_i^{a_i}$ is a generator of $\mathrm{Ann}(M)$, where p_i are irreducible elements of $k[t]$. Prove that M can be decomposed as a direct sum of submodules, $M = M_1 \oplus \cdots \oplus M_\ell$, such that $\mathrm{Ann}(M_i) = (p_i^{a_i})$.

(c) State an analogous statement over $\mathbb{Z}$ instead of $k[t]$. Is it true?

(d) State (carefully) an analogous statement for $R = \mathbb{Z}[\sqrt{-5}]$. Is it true?

5. (10 points) (a) Let p be a prime and let $H = \mathbb{Z}/p\mathbb{Z} \times \cdots \times \mathbb{Z}/p\mathbb{Z}$ be the product of n factors. Prove that the group $\mathrm{Aut}(H)$ of automorphisms of H is isomorphic to $\mathrm{GL}_n(\mathbb{F}_p)$, where $\mathbb{F}_p$ is the field of p elements.
   (b) Find the order of $\mathrm{GL}_2(\mathbb{F}_p)$. (Bonus: find the order of $\mathrm{GL}_n(\mathbb{F}_p)$.)
   (c) Assuming that $|\,\mathrm{GL}_3(F_2)| = 168$, prove that there exists a group of order 56 with a non-normal subgroup of order 7.

6. (7 points) Let $p(x) = x^3 - 4x + 2$.
   (a) Is this polynomial irreducible over $\mathbb{Q}$?
   (b) Find the degree of its splitting field over $\mathbb{Q}$.