The University of British Columbia
Department of Mathematics
Qualifying Examination—Linear Algebra and Differential Equations

January 13, 2024

Linear Algebra

1. (10 points) Consider the linear system

$$Ax = b \qquad \text{with } A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

(a) [5 points] For what vectors b will this system have at least one solution? Find an orthogonal basis for this set.

(b) [5points] Suppose $b = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$. Is a solution possible? If so, characterize all possible solutions. If not solution is possible, find the best approximation, i.e. an x that minimizes $|Ax - b|$. If this approximate solution is not unique, characterize all possible optimal solutions.

2. (10 points) Consider the map

$$D : f(t) \to ct\frac{df}{dt} + \frac{d^2 f}{dt^2}$$

D defines a linear map of the space L_n of polynomials $f(t)$ of degree $\leq n$ into itself, and $c \in \mathbb{R}$ is a parameter.

(a) [3 points] What is the rank of D, depending on c?

(b) [4 points] What are the eigenvalues of D, depending on c?

(c) [3 points] Find three non-zero eigenvectors of D for $c \neq 0$.

3. (10 points) Recall that for a symmetric matrix A we write $A > 0$ to mean that A is positive definite, i.e. $x^T Ax > 0$ for all vectors x. If A and B are $n \times n$ real symmetric matrices such that $A - B$ is positive

definite, then we write $A > B$.

(a) [4 points] If $A > 0$, show that there exists a symmetric matrix $L > 0$ such that $L^2 = A$.

(b) [4 points] If $A, B > 0$ and $A > I$, where I is the identity matrix, show that $A^{-1} < I$.

(c) [2 points] f $A, B > 0$ and $A > B$, show that $A^{-1} < B^{-1}$.

Differential Equations

4. (10 points) A guitar string is plucked at the center so that the initial shape of the sting is given by

$$f(x) = \begin{cases} x & \text{for } x \in [0, 1)\ , \\ 2 - x & \text{for } x \in [1, 2]\ . \end{cases}$$

Note that $f(x)$ is only defined here on the interval $[0, 2]$. The deflection of the string, $u(x,t)$, satisfies the PDE

$$u_{tt} = c^2 u_{xx}$$

with boundary and initial conditions

$$u(0,t) = 0, \qquad u(1,t) = 0, \qquad u(x,0) = f(x), \qquad u_t(x,0) = 0.$$

(a) Using Separation of Variables and the formula given at the end of the exam, solve the IBVP. Your calculations and explanations should be complete. Assume the reader is mathematical sophisticated but has no previous knowledge of the method. No explanation of where the formula at the end of the exam comes from nor why the sum or its derivatives converge is required.

(b) Use the trig identity $\sin(a)\cos(b) = \frac{1}{2}(\sin(a+b) + \sin(a-b))$ to derive a simpler expression for the solution that consists of a sum of two functions that translate in time in opposite directions at a speed c. How do these two function relate to f?

5. (10 points) Consider the differential equation

$$\frac{d^2y}{dt^2} + ay = k(t)$$

where $a > 0$ and $k(t)$ is the alternating square wave with period 4 and defined on $[0, 4)$ by

$$k(t) = \begin{cases} 1 & \text{for } t \in [0,1) \cup [3,4)\ , \\ -1 & \text{for } t \in [1,3)\ . \end{cases}$$

Find the general solution to the ODE. Be sure to account for all possible values of $a > 0$. The Fourier series at the end of the exam should prove useful for this problem as well (*Hint:* The derivative of $f(t)$ is $k(t)$).

6. (10 points) Consider the non-dimensional system of differential equations that models a predator-prey interaction with the prey having logistic growth in the absence of predators:

$$x' = ax(1-x) - bxy,$$
$$y' = bxy - y,$$

where a and b are positive parameters.

(a) Find all steady states for the system.

(b) Classify each steady state (e.g. stable node, unstable spiral etc.). The classification may depend on the parameter values. You should classify them whether or not they correspond to a meaningful biological steady state. If you are not familiar with a name for any classification, you can sketch a picture of the solution curves near the steady state.

Formulas

$$\begin{aligned} f(x) &= \begin{cases} x & \text{for } x \in [0,1) \\ 2-x & \text{for } x \in [1,2) \end{cases} \\ &= \sum_{n=1}^{\infty} \frac{8}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right) \sin\left(\frac{n\pi x}{2}\right) \qquad \text{for } x \in [0,2) \end{aligned}$$