Qualifying Exam Problems: Analysis
(September 9, 2014)

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1. (10 points) Compute the limit

$$\lim_{n\to\infty}\int_0^{n^{\frac{1}{3}}}(1-\frac{x^2}{n})^n dx.$$

   Write your answer in the form of a definite integral, justifying each step in your calculation. Then evaluate the integral.

2. (10 points) Let $f:[0,1]\to\mathbb{R}$ be a bounded measurable function. Let $g:[-2,2]\to\mathbb{R}$ be Lebesgue integrable. Show that

$$\lim_{h\to 0}\int_0^1 f(x)g(x+h)dx=\int_0^1 f(x)g(x)dx.$$

3. (10 points) Let $k\geq 1$ be an integer. Suppose $f:[0,k]\to\mathbb{R}$ is a continuous function with $f(0)=f(k)$. Show that there exists at least k different pairs of x_1,x_2, such that $f(x_2)=f(x_1)$ and x_2-x_1 is an integer.

4. (10 points) Use branch cut and contour integrals to evaluate the integral

$$I=\int_0^\infty\frac{(\ln x)^2}{1+x^2}\,dx$$

5. (a) (5 points) How many solutions are there to the equation

$$z^7+2=e^{-z}$$

   in the right-hand half-plane where $Re(z)>0$? Justify your answer.

   (b) (5 points) Find an explicit analytic function ϕ mapping the sector $\{Re^{i\theta}|r>0,-\frac{\pi}{4}<\theta<\frac{\pi}{4}\}$ onto the open unit disk $\{|z|<1\}$.

6. Let f be an entire function and $a,b\in\mathbb{C}, a\neq b$.

   (a) (5 points) Evaluate the integral $\int_{|z|=R}\frac{f(z)}{(z-a)(z-b)}dz$ for $R>|a|$ and $R>|b|$.

   (b) (5 points) Assume that $|f(z)|\leq\sqrt{1+|z|}$ for all $z\in\mathbb{C}$. Use part (a) to show that f must be a constant.