Qualifying Exam Problems: Algebra
(September 9, 2014)

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1. (10 points) Let

$$A = \begin{pmatrix} -11 & 9 \\ -30 & 22 \end{pmatrix}$$

   Find A^{2014}.

2. Let $n \geq 2$ be an integer, let $M_{n,n}(\mathbb{R})$ be the set of all n-by-n matrices with real entries, let $B \in M_{n,n}(\mathbb{R})$ and let $f_B : M_{n,n}(\mathbb{R}) \longrightarrow M_{n,n}(\mathbb{R})$ be given by

$$f_B(A) = AB - BA$$

   for each $A \in M_{n,n}(\mathbb{R})$.

   (a) (2 points) Show that f_B is a linear map.

   (b) (3 points) If B has distinct eigenvalues, show that $\dim \ker(f_B) \geq n$.

   (c) (5 points) If $n = 2$ and B is not diagonalizable, find $\dim \ker(f_B)$.

3. (a) (1 point) Let $n \geq 2$ be an integer, let $A, B \in M_{n,n}(\mathbb{R})$ and let $\lambda \in \mathbb{C}$. If A is invertible, prove that $\lambda \cdot I_n - AB$ is invertible if and only if $\lambda \cdot A^{-1} - B$ is invertible.

   (b) (2 points) Let $n \geq 2$ be an integer, let $A, B \in M_{n,n}(\mathbb{R})$ and let $\lambda \in \mathbb{C}$. If A is invertible, prove that $\det(\lambda \cdot I_n - AB) = \det(\lambda \cdot I_n - BA)$.

   (c) (3 points) Let $n \geq 2$ be an integer, and let $A, B \in M_{n,n}(\mathbb{R})$. Show that $\lambda \in \mathbb{C}$ is an eigenvalue of AB if and only if it is an eigenvalue of BA.

   (d) (4 points) Let $C \in M_{2,3}(\mathbb{R})$ and $D \in M_{3,2}(\mathbb{R})$ such that

$$DC = \begin{pmatrix} 2 & -1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 5 \end{pmatrix}$$

   Find $\det(CD)$.

4. (10 points) Let $Z \subset G$ be the center of a group G and suppose that G/Z is cyclic. Prove that G is Abelian.

5. (a) (4 points) Determine the minimal polynomial for $\alpha = \sqrt{3} + \sqrt{5}$ over the field $\mathbb{Q}$.

   (b) (2 points) Determine the minimal polynomial for $\alpha = \sqrt{3} + \sqrt{5}$ over the field $\mathbb{Q}(\sqrt{5})$.

   (c) (2 points) Determine the minimal polynomial for $\alpha = \sqrt{3} + \sqrt{5}$ over the field $\mathbb{Q}(\sqrt{10})$.

   (d) (2 points) Determine the minimal polynomial for $\alpha = \sqrt{3} + \sqrt{5}$ over the field $\mathbb{Q}(\sqrt{15})$.

6. (a) (2 points) Let G be a group of prime order p. Show that the order of $\mathrm{Aut}(G)$, the automorphism group of G, is $p - 1$.

   (b) (2 points) Let G be a group and let $N \subset G$ be a normal subgroup. Show that conjugation induces a homomorphism $\phi : G \to \mathrm{Aut}(N)$.

   (c) (3 points) Show that a group G of order 15 is cyclic.

   (d) (3 points) Show that if the order of a group G is 255, then G is cyclic. Hint: Find the number of Sylow 17-subgroups and use the results of parts (a), (b), and (c).