Qualifying Exam Problems: Linear Algebra and Differential Equations
(September 9, 2014)

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1. (10 points) Let
$$A = \begin{pmatrix} -11 & 9 \\ -30 & 22 \end{pmatrix}$$
Find A^{2014}.

2. Let $n \geq 2$ be an integer, let $M_{n,n}(\mathbb{R})$ be the set of all n-by-n matrices with real entries, let $B \in M_{n,n}(\mathbb{R})$ and let $f_B : M_{n,n}(\mathbb{R}) \longrightarrow M_{n,n}(\mathbb{R})$ be given by
$$f_B(A) = AB - BA$$
for each $A \in M_{n,n}(\mathbb{R})$.
   (a) (2 points) Show that f_B is a linear map.
   (b) (3 points) If B has distinct eigenvalues, show that $\dim \ker(f_B) \geq n$, where $\ker(f_B)$ is the kernel (or nullspace) of f_B.
   (c) (5 points) If $n = 2$ and B is not diagonalizable, find $\dim \ker(f_B)$.

3. (a) (1 point) Let $n \geq 2$ be an integer, let $A, B \in M_{n,n}(\mathbb{R})$ and let $\lambda \in \mathbb{C}$. If A is invertible, prove that $\lambda \cdot I_n - AB$ is invertible if and only if $\lambda \cdot A^{-1} - B$ is invertible.
   (b) (2 points) Let $n \geq 2$ be an integer, let $A, B \in M_{n,n}(\mathbb{R})$ and let $\lambda \in \mathbb{C}$. If A is invertible, prove that $\det(\lambda \cdot I_n - AB) = \det(\lambda \cdot I_n - BA)$.
   (c) (3 points) Let $n \geq 2$ be an integer, and let $A, B \in M_{n,n}(\mathbb{R})$. Show that $\lambda \in \mathbb{C}$ is an eigenvalue of AB if and only if it is an eigenvalue of BA.
   (d) (4 points) Let $C \in M_{2,3}(\mathbb{R})$ and $D \in M_{3,2}(\mathbb{R})$ such that
$$DC = \begin{pmatrix} 2 & -1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 5 \end{pmatrix}$$
   Find $\det(CD)$.

4. Consider the first order system
$$\dot{\mathbf{x}}(t) = \begin{bmatrix} \alpha & -1 \\ 1 & \alpha \end{bmatrix} \mathbf{x}(t)$$
   (a) (4 points) If $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are solutions with $\mathbf{x}(0)$ and $\mathbf{y}(0)$ linearly independent, show that $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are linearly independent for all t.
   (b) (4 points) Find solutions $\mathbf{x}(t)$ and $\mathbf{y}(t)$ with $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{y}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
   (c) (2 points) If $\mathbf{x}(t)$ is a solution with $\mathbf{x}(0)$ in the first quadrant (i.e., $x_1(0) > 0$ and $x_2(0) > 0$), how many times has $\mathbf{x}(t)$ crossed the positive x_1 axis when $t = 9\pi$?

5. Consider the initial value problem
$$\ddot{x}(t) = -V'(x(t)), \quad x(0) = a, \quad \dot{x}(0) = b \tag{1}$$
where $V : \mathbb{R} \to \mathbb{R}$ is a smooth function.

(a) (2 points) Find a function $H : \mathbb{R}^2 \to \mathbb{R}$ and suitable initial conditions so that the (1) is equivalent to the first order system

$$\begin{aligned} \dot{x}(t) &= \frac{\partial H}{\partial p}(x(t), p(t)) \\ \dot{p}(t) &= -\frac{\partial H}{\partial x}(x(t), p(t)) \end{aligned} \tag{2}$$

(b) (2 points) Show that $H(x(t), p(t))$ is constant along the trajectories of (2).

(c) (3 points) Suppose that $x = 0$ is a strict local minimum of $V(x)$. Show that $(x, p) = (0, 0)$ is an equilibrium point. Explain why $(x, p) = (0, 0)$ is stable but not asymptotically stable.

(d) (3 points) Now suppose $V(x) = -x^4/2$ and let $(x(t), p(t))$ be the solution of (2) with $x(0) = 0$ and $p(0) = 1$. Show that $x(t)$ reaches infinity in finite time with the following steps. First show that $\dot{x} \geq 0$ for all t. Then use part (b) to write down a first order equation satisfied by $x(t)$. Using this equation, write down an expression for $t(x)$, the inverse function to $x(t)$. Then show that $t(\infty) < \infty$

6. (a) (4 points) Use separation of variables (Fourier series) to solve the Cauchy problem

$$u_{tt} = \alpha^2 u_{xx}$$

for $t \geq 0$ and for $x \in [0, 4]$, with boundary conditions

$$u(0, t) = u(4, t) = 0,$$

and with initial conditions

$$u(x, 0) = u_0(x) = \begin{cases} 0 & 0 \leq x \leq 1 \\ 1 & 1 < x < 3 \\ 0 & 3 \leq x \leq 4 \end{cases}$$

and

$$u_t(x, 0) = 0.$$

(b) (3 points) Now use d'Alembert's formula to solve the Cauchy problem

$$v_{tt} = \alpha^2 v_{xx}$$

for $t \geq 0$ and for $x \in \mathbb{R}$, with initial conditions

$$v(x, 0) = v_0(x) = \begin{cases} 0 & -\infty < x \leq 1 \\ 1 & 1 < x < 3 \\ 0 & 3 \leq x < \infty \end{cases}$$

and

$$v_t(x, 0) = 0.$$

(c) (3 points) For what values of $t \geq 0$ do the solutions of parts (a) and (b) agree (for $0 \leq x \leq 4$)? Write down the explicit form of $v(x, 1/(2\alpha))$ for $0 \leq x \leq 4$. What is the Fourier series representation of this function?