Mathematics Qualifying Exam

University of British Columbia

September 2, 2010

Part I: Real and Complex Analysis (Pure and Applied Exam)

1. a) Find all continuously differentiable vector fields $\mathbf{F}(x, y, z)$ from $\mathbb{R}^3$ to $\mathbb{R}^3$ with the property that $\oint_C \mathbf{F}(x, y, z) \times d\mathbf{r} = \mathbf{0}$ for all closed curves C in $\mathbb{R}^3$.

   b) Show that $\displaystyle\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$.

2. Consider $T : \mathbb{R}^n \to \mathbb{R}^n$ where $\mathbb{R}^n$ is the n-dimensional Euclidean space with the standard Euclidean metric. Suppose that for all k large enough, T^k satisfies:
$$|T^k x - T^k y| \leq r|x - y|$$
where $r \in (0, 1)$. Show that T has a unique fixed point.

3. a) Show that $I = \displaystyle\int_0^\infty \frac{\sin x}{x}\,dx$ converges.

   b) Evaluate I.

   c) For which real values of p and q does $\displaystyle\int_0^\infty \frac{\sin x}{|1-x|^q\, x^p}\,dx$ converge? Justify your answer.

4. Consider the function $f(z) = e^{\frac{z+1}{z-1}}$. Find all z for which

   a) $|f(z)| = 1$;

   b) $|f(z)| < 1$.

5. Let $f(z) = \dfrac{1}{(2z-1)(z-2)}$.

   a) Give the first three nonzero terms for the Laurent expansion of $f(z)$ about $\frac{1}{2}$.

   b) Give the first three nonzero terms for the Laurent expansion of $f(z)$ about 0, valid for small $|z|$. Give the region of convergence for the full expansion.

   c) Give the first three nonzero terms for the Laurent expansion of $f(z)$ about 0, valid for large $|z|$. Give the region of convergence for the full expansion

   d) Compute $f''(0)$.

   e) Evaluate $\oint_{|z|=\frac{3}{2}} f(z)\,dz$.

   f) Evaluate $\oint_{|z|=79} f(z)\,dz$.

6. Let H^+ be the upper half plane. For $z \in \mathsf{H}^+$, define the function
$$h_z(\zeta) = \frac{1}{2\pi i}\left(\frac{1}{\zeta - z} - \frac{1}{\zeta - \bar{z}}\right) \quad \text{for all } \zeta \in \mathsf{H}^+,$$
then h_z is analytic on H^+ except for a simple pole at z. Suppose that f is a bounded analytic function on $\mathsf{H}^+ \cup \mathbb{R}$. Prove that
$$\int_{-\infty}^{\infty} f(t)\,h_z(t)\,dt = f(z).$$

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Part II: Linear Algebra and Algebra (Pure Exam)

1. A matrix $A \in \mathbb{R}^{n\times n}$ is called positive definite if $x^\top Ax > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$. Show:

   a) If $A \in \mathbb{R}^{n\times n}$ is symmetric and positive definite, then all its eigenvalues are positive.

   b) If $A \in \mathbb{R}^{n\times n}$ is positive definite and $X \in \mathbb{R}^{n\times k}$ has rank k, then $X^\top AX$ is also positive definite.

   c) Let A be the symmetric matrix
   $$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}.$$
   Verify that if A is positive definite, then
   $$a_{11} > 0, \qquad a_{22} > 0, \qquad |a_{12}| < \frac{a_{11} + a_{22}}{2}.$$

2. The trace of a matrix $A \in \mathbb{R}^{n\times n}$ is defined by $\mathrm{tr}(A) = \sum_{i=1}^{n} a_{ii}$.

   a) Show that if $A \in \mathbb{R}^{n\times n}$ and $A' \in \mathbb{R}^{n\times n}$ are similar, then $\mathrm{tr}(A) = \mathrm{tr}(A')$.

   b) For any matrix $A \in \mathbb{R}^{n\times n}$, there holds $\det(e^A) = e^{\mathrm{tr}(A)}$. Prove this identity in the case where A is diagonalizable.

3. Let A be a square matrix with all diagonal entries equal to 2, all entries directly above or below the main diagonal equal to 1, and all other entries equal to 0. Show that every eigenvalue of A is a real number strictly between 0 and 4.

4. If a is an element of a finite field with 9 elements, show that $a^9 = a$.

5. Let p be an irreducible quartic polynomial over the field $\mathbb{Q}$ of rational numbers, and let W be the Galois group of p. If p has exactly two real roots, show that W is of order 24 or 8.

6. How many automorphisms does the group $(\mathbb{Z}/2\mathbb{Z})^3$ possess?