Applied Mathematics Qualifying Exam
University of British Columbia
September 4, 2009.

Part I

1. (a) Let
$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$
and verify that for all real numbers x_1, y_1, x_2, y_2 we have
$$(x_1 I + y_1 J)(x_2 I + y_2 J) = (x_1 x_2 - y_1 y_2) I + (x_1 y_2 + x_2 y_1) J.$$
(b) Find A^n for any integer n, if
$$A = \begin{bmatrix} 1/4 & -\sqrt{3}/4 \\ \sqrt{3}/4 & 1/4 \end{bmatrix}.$$

2. Prove that the equation $XY - YX = I_n$ has no solution (where X, Y are unknown real $n \times n$-matrices, and I_n is the identity matrix).

3. A nonzero matrix A is called nilpotent if there exists a positive integer n such that $A^n = 0$. Two matrices A and B are called similar if they can be obtained from one another by a change of basis.

   a) Prove that if A is nilpotent and B is similar to A, then B is also nilpotent.

   b) Find a set of representatives of all equivalence classes of nilpotent 3×3-matrices with complex entries, where we declare two matrices equivalent if they are similar. (You may want to solve this question for 2×2-matrices first).

4. Define the Fourier transform pair to be:
$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} dx \qquad \text{and} \qquad f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx} dx.$$
(a) Use contour integration to calculate the Fourier transform $\hat{f}(k)$ for
$$f(x) = \frac{1}{(x^2 + a^2)^2},$$
where $a \in \mathbb{R}$ is a constant.

(b) Calculate the inverse Fourier transform $f(x)$ for
$$\hat{f}(k) = \frac{1}{(k^2 + a^2)^2},$$
where $a \in \mathbb{R}$ is a constant.

5. Use a keyhole-shaped contour to evaluate the integral

$$I = \int_0^\infty \frac{dx}{\sqrt{x}(x^2+1)}.$$

6. Let D be the circle of radius 4 centred at the point $(0,5)$ in the $x-y$ plane. Find a function $\phi(x,y)$ that satisfies the following restrictions:

   - ϕ is harmonic in the upper half-plane exterior to D;
   - $\phi = 1$ on D;
   - $\phi = 0$ on the x-axis.

   Hint: Consider a conformal map of the form $w = \frac{z+\alpha}{z+\beta}$.

Part II

1. Suppose f is a continuous function on $\mathbb{R}$ such that $|f(x) - f(y)| \geq |x - y|$ for all x and y. Show that the range of f is all of $\mathbb{R}$.

2. For every $a \in \mathbb{R}$, determine whether the integral

$$\iint_D (x^4 + y^2)^a \, dA$$

is finite, where D is the square $\{(x, y) \mid -1 \leq x \leq 1, -1 \leq y \leq 1\}$.

3. Let S be the finite solid region bounded by the plane $z = 0$ and the surface $z = 1 - x^2 - y^2$. Find the flux of the vector field $\mathbf{V} = xy\mathbf{i} + xz\mathbf{j} + (1 - z - yz)\mathbf{k}$ outward through the surface of S.

4. (a) Find the general solution of the homogeneous linear system

$$x' = x + y, \quad y' = y.$$

(b) Solve the initial value problem

$$x' = x + y + e^t\sqrt{1+t}, \quad y' = y + \frac{e^t}{1+t^2}, \qquad t > -1,$$

$$x(0) = 0, \quad y(0) = 1.$$

5. Consider the system of ordinary differential equations in the (x, y) plane

$$x' = (a - \pi^2)x - 3x^3 - 6xy^2, \quad y' = (a - 4\pi^2)y - 6x^2y - 3y^3, \quad (1)$$

where a is a real constant. Throughout this question, assume that $0 < a < 7\pi^2$.

(a) Find all equilibria (i.e. critical points or constant solutions or steady states), and determine the linearized stability and type of each equilibrium. You may wish to consider different cases.

(b) Let $V(x, y) = \frac{1}{2}(x^2 + y^2)$. Show that for all solutions $(x(t), y(t))$ of (1) with sufficiently large distance from the origin, the expression $V(x(t), y(t))$ is a decreasing function of t. Discuss the behaviour as $t \to \infty$, of solutions of (1).

6. (a) Solve the initial boundary value problem

$$u_t = u_{xx} + au, \qquad 0 < x < 1, \quad t > 0,$$

$$u(0, t) = 0, \quad u(1, t) = 0, \qquad t > 0,$$

$$u(x, 0) = g(x), \qquad 0 \leq x \leq 1,$$

where a is a positive constant, and $g(x)$ is a continuous function defined for $0 \leq x \leq 1$, with $g(0) = g(1) = 0$. Describe the solution's behaviour as $t \to \infty$. You may wish to consider different cases of a and $g(x)$.

(b) Let $t > 0$ be fixed, and let $\hat{f}(k) = \int_{-\infty}^{\infty} e^{ikx} e^{-x^2/(4t)}\,dx$, $-\infty < k < \infty$. Show that $\hat{f}$ satisfies the differential equation $\hat{f}' = -2tk\hat{f}$, then solve this differential equation to find an explicit formula for $\hat{f}(k)$ in terms of elementary functions. You may use the fact that $\int_{-\infty}^{\infty} e^{-\alpha x^2}\,dx = \sqrt{\pi/\alpha}$ for any constant $\alpha > 0$.

(c) Solve the initial boundary value problem

$$u_t = u_{xx} + au, \qquad -\infty < x < \infty, \quad t > 0,$$

$$\lim_{x\to\pm\infty} u(x,t) = 0, \qquad \lim_{x\to\pm\infty} u_x(x,t) = 0, \qquad t > 0,$$

$$u(x,0) = g(x), \qquad -\infty < x < \infty,$$

where a is a positive constant, and $g(x)$ is a continuous function defined for $-\infty < x < \infty$, with $\lim_{x\to\pm\infty} g(x) = \lim_{x\to\pm\infty} g'(x) = 0$. Express the solution as a single integral over a spatial variable. Describe the solution's behaviour as $t \to \infty$. You may wish to consider different cases of a and $g(x)$.