Pure Mathematics Qualifying Exam

University of British Columbia

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Part I

1. Suppose f is a continuous real-valued function on $[0, 1]$. Show that

$$\int_0^1 f(x)x^2\,dx = \frac{1}{3}f(\xi)$$

for some $\xi \in [0, 1]$.

2. Let $\varphi_1, \varphi_2, \cdots, \varphi_n, \cdots$ be non-negative continuous functions on $[0, 1]$ such that the limit

$$\lim_{n\to\infty} \int_0^1 x^k \varphi_n(x)\,dx$$

exists for every $k = 0, 1, 2, \cdots$. Show that the limit

$$\lim_{n\to\infty} \int_0^1 f(x)\varphi_n(x)\,dx$$

exists for every continuous function f on $[0, 1]$.

3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ have directional derivatives in all directions at the origin. Is f differentiable at the origin? Prove or give a counter-example.

4. Consider the $n \times n$ matrix with a 7 in every entry of the first p rows and 4 in every entry of the last $n - p$ rows. Find its eigenvalues and eigenvectors.

5. Recall that matrices A and B are called **similar** provided that there exists an invertible matrix P such that $A = PBP^{-1}$. Also recall that **det** and **tr** are preserved under the similarity transformation $B \to PBP^{-1}$. For a and ϵ real, define the matrix:

$$A_\epsilon = \begin{pmatrix} a & \epsilon \\ 0 & a \end{pmatrix}.$$

   (a) Show that the family of matrices $\mathcal{F} = \{A_\epsilon : \epsilon \neq 0\} \cup \{A_\epsilon^T : \epsilon \neq 0\}$ are all similar to one another. Note: the superscript ^{T} denotes matrix transpose.

   (b) Show that the following classes of real 2×2 matrices are each a distance 0 away from the family $\mathcal{F}$:

   – the class of matrices with one eigenvalue with geometric multiplicity two,

   – the class of matrices with distinct real eigenvalues,

   – the class of matrices with non-real complex eigenvalues,

   where distance is defined using the max norm $(||A||_{max} = \max\{|a_{ij}|\})$.

6. Let L be a linear transformation from polynomials of degree less than or equal to two to the set of 2×2 matrices ($L : \mathcal{P}^2(\mathbb{R}, \mathbb{R}) \to \mathcal{M}(2, 2)$) given by

$$L(a_0 + a_1 x + a_2 x^2) = \begin{pmatrix} a_0 + a_2 & a_0 + a_1 \\ a_0 + a_2 & a_0 + a_1 \end{pmatrix}$$

   (a) Verify that this transformation is linear.

   (b) Find the matrix that represents the linear transformation $\mathcal{L}$ with respect to the bases

$$\begin{aligned} \mathcal{V} &= \{1, x, x^2\} \\ \mathcal{W} &= \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} \end{aligned}$$

   which are bases for $\mathcal{P}^2(\mathbb{R}, \mathbb{R})$ and $\mathcal{M}(2, 2)$ respectively.

   (c) Find bases for the nullspace $\mathcal{N}(L)$ and range $\mathcal{R}(L)$.

Part II

1. Find the Laurent series expansion of $f(z) = \dfrac{4}{(1+z)(3-z)}$ around $z_0 = 0$ in the annulus $1 < |z| < 3$.

2. Let $f(z) = \left(\dfrac{\sin(3z)}{z^2} - \dfrac{3}{z}\right) \cdot \left(\dfrac{z+1}{z+2}\right) \cdot \exp\left(\dfrac{1}{z-5}\right)$.

   (a) Find and classify all singularities of f.

   (b) Evaluate $I = \int_\Gamma f(z)dz$ where Γ is the positively oriented triangular loop with vertices at $v_1 = -1 - i$, $v_2 = 1 - i$, and $v_3 = i$.

3. Compute the integral
$$\int_0^{2\pi} \frac{1}{2 + \cos(x)} dx.$$
   [Convert to an integral on the unit circle via a substitution, then use residue theory.]

4. Suppose E is an algebraic field extension of a field F. Suppose R is a subring of E which contains F. Show that R is a field.

5. Let $\mathbf{GL}_3(\mathbb{Z})$ denote the group of invertible 3×3-matrices with integer entries. Show that $\mathbf{GL}_3(\mathbb{Z})$ has no element of order 7.

6. Show that any group of order 765 is abelian.