Applied Mathematics Qualifying Exam

University of British Columbia

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Part I

1. Suppose f is a continuous real-valued function on $[0, 1]$. Show that

$$\int_0^1 f(x)x^2\,dx = \frac{1}{3}f(\xi)$$

for some $\xi \in [0, 1]$.

2. Consider the vector field $\mathbf{F}(x, y, z) = (\sin(x) + y^2)\mathbf{i} + (\cos(y) + z^2)\mathbf{j} + (e^{-z} + x^2)\mathbf{k}$.

   (a) Compute curl $\mathbf{F}$ and div $\mathbf{F}$.

   (b) Is $\mathbf{F}$ conservative? Justify your answer.

   (c) Can $\mathbf{F}$ be written as the curl of another vector field $\mathbf{G}$? Justify your answer.

   (d) Let C be the curve of intersection of the cylinder $x^2 + y^2 = 2x$ and plane $z = x$ oriented counterclockwise as viewed from above. Denote by S the part of the plane $z = x$ that is bounded by C and oriented upward.

      i. Parametrize C.

      ii. Parametrize S.

      iii. State Stokes' Theorem and use it to calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$.
      [*Hint*: The following might be useful: $\cos^2(\theta) = \frac{1}{2}(1 + \cos(2\theta))$.]

3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ have directional derivatives in all directions at the origin. Is f differentiable at the origin? Prove or give a counter-example.

4. Consider the $n \times n$ matrix with a 7 in every entry of the first p rows and 4 in every entry of the last $n - p$ rows. Find its eigenvalues and eigenvectors.

5. Recall that matrices A and B are called **similar** provided that there exists an invertible matrix P such that $A = PBP^{-1}$. Also recall that **det** and **tr** are preserved under the similarity transformation $B \to PBP^{-1}$. For a and ϵ real, define the matrix:

$$A_\epsilon = \begin{pmatrix} a & \epsilon \\ 0 & a \end{pmatrix}.$$

   (a) Show that the family of matrices $\mathcal{F} = \{A_\epsilon : \epsilon \neq 0\} \cup \{A_\epsilon^T : \epsilon \neq 0\}$ are all similar to one another. Note: the superscript ^{T} denotes matrix transpose.

   (b) Show that the following classes of real 2×2 matrices are each a distance 0 away from the family $\mathcal{F}$:

      – the class of matrices with one eigenvalue with geometric multiplicity two,
      – the class of matrices with distinct real eigenvalues,
      – the class of matrices with non-real complex eigenvalues,

   where distance is defined using the max norm ($||A||_{max} = \max\{|a_{ij}|\}$).

6. Let L be a linear transformation from polynomials of degree less than or equal to two to the set of 2×2 matrices ($L : \mathcal{P}^2(\mathbb{R}, \mathbb{R}) \to \mathcal{M}(2, 2)$) given by

$$L(a_0 + a_1 x + a_2 x^2) = \begin{pmatrix} a_0 + a_2 & a_0 + a_1 \\ a_0 + a_2 & a_0 + a_1 \end{pmatrix}$$

   (a) Verify that this transformation is linear.

   (b) Find the matrix that represents the linear transformation $\mathcal{L}$ with respect to the bases

$$\begin{aligned} \mathcal{V} &= \{1, x, x^2\} \\ \mathcal{W} &= \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} \end{aligned}$$

   which are bases for $\mathcal{P}^2(\mathbb{R}, \mathbb{R})$ and $\mathcal{M}(2, 2)$ respectively.

   (c) Find bases for the nullspace $\mathcal{N}(L)$ and range $\mathcal{R}(L)$.

Part II

1. Find the Laurent series expansion of $f(z) = \dfrac{4}{(1+z)(3-z)}$ around $z_0 = 0$ in the annulus $1 < |z| < 3$.

2. Let $f(z) = \left(\dfrac{\sin(3z)}{z^2} - \dfrac{3}{z}\right) \cdot \left(\dfrac{z+1}{z+2}\right) \cdot \exp\left(\dfrac{1}{z-5}\right)$.

   (a) Find and classify all singularities of f.

   (b) Evaluate $I = \int_\Gamma f(z)dz$ where Γ is the positively oriented triangular loop with vertices at $v_1 = -1-i$, $v_2 = 1-i$, and $v_3 = i$.

3. Compute the integral
$$\int_0^{2\pi} \frac{1}{2+\cos(x)} dx.$$
   [Convert to an integral on the unit circle via a substitution, then use residue theory.]

4. Glycolysis is the enzymatic process by which glucose is broken down for the purpose of extracting energy. Consider the following simplified model of glycolysis proposed by Schnakenberg (1979):
$$\begin{aligned} \frac{dx}{dt} &= x^2y - x \\ \frac{dy}{dt} &= a - x^2y. \end{aligned}$$
   As the parameter a varies from $-\infty$ to ∞, the structure of solutions near the steady state of the system changes. Determine the sequence of steady state classifications for increasing values of a (e.g. stable node-to-saddle-to-unstable node).

5. The voltage across the capacitor in an RLC circuit being driven by an oscillatory potential is given by the solution of the equation:
$$LC\frac{d^2v}{dt^2} + RC\frac{dv}{dt} + v = V_0\cos(wt)$$
   where L, R, C, V_0 and w, all non-negative, are physical parameters determined by the circuit components.

   (a) Calculate the so-called natural frequency of the circuit, when both $R = 0$ and $V_0 = 0$?

   (b) Provide expressions for the change of variables, $t \to \tau$ and $v \to y$, that simplifies the equation to: $y'' + ay' + by = \cos(t)$ where $'$ denotes derivative with respect to τ.

   (c) Calculate the general solution.

   (d) Show that the solution converges to $v(t) = A\cos(t-\delta)$ as $t \to \infty$ where $A = 1/\sqrt{a^2 + (b-1)^2}$ and δ satisfies $\cos(\delta) = (b-1)A$.

   (e) Returning to the original variables and parameters, what forcing frequency w maximizes the amplitude of the response?

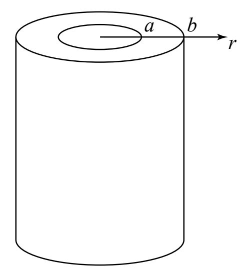
6. Consider the physical problem of a long metal cylinder with annular cross-section. The temperature in Kelvin on the interior of the metal is described by the equation

$$u_t = D\left(u_{rr} + \frac{1}{r}u_r\right)$$

where r is the radial coordinate measured from the middle of the cylinder and we have assumed that u does not vary along the height of the cylinder. The inner and outer surfaces of the cylinder, located at $r = a$ and $r = b$ respectively, are treated such that the following boundary conditions apply:

$$-Du_r(a,t) = \alpha, \quad -Du_r(b,t) = \beta$$

where α and β are both positive. Initially, u is given by $u(r,0) = f(r)$.



(a) Provide a physical interpretation of the boundary conditions. What are the units on α and β?

(b) What condition on α and β must be satisfied for a steady state solution to exist? Assume it is satisfied and calculate the steady state.

(c) If you were to solve the time-dependent problem by an eigenfunction decomposition, what equation would the eigenfunctions satisfy? You do not need to solve the equation.