# Qualifying Exam
## Pure Mathematics Part I.

**Problem 1.**
Prove that the matrix

$$\begin{pmatrix} 0 & 5 & 1 & 0 \\ 5 & 0 & 5 & 0 \\ 1 & 5 & 0 & 5 \\ 0 & 0 & 5 & 0 \end{pmatrix}$$

has two positive and two negative eigenvalues (counting multiplicities).

**Problem 2.**
Prove that there exists only one automorphism of the field of real numbers; namely the identity automorphism.

**Problem 3.**
Let $G$ be the abelian group on generators $x$, $y$ and $z$, subject to the relations

$$\begin{array}{ccccccc} 32x & + & 33y & + & 26z & = & 0 \\ 29x & + & 31y & + & 27z & = & 0 \\ 27x & + & 28y & + & 26z & = & 0 \end{array}$$

How many elements does $G$ have? Is $G$ cyclic?

**Problem 4.**
Let $x_0 = 0$ and

$$x_{n+1} = \frac{1}{2 + x_n}$$

for $n = 0, 1, 2, \ldots$. Prove that $x_\infty = \lim_{n\to\infty} x_n$ exists and find its value.

**Problem 5.**
Let $\mathbf{A}$ be the $n \times n$ matrix with all diagonal entries $s$ and all off-diagonal entries $t$. For which complex values of $s$ and $t$ is this matrix not invertible? For each of these values, describe the nullspace of $\mathbf{A}$ (including its dimension).

**Problem 6.**
Evaluate

$$\int_{-\infty}^{\infty} \frac{\sin 3x dx}{x^2 + 2x + 3}.$$

# Qualifying Exam
## Pure Mathematics Part II.

**Problem 7.**
Determine the Jordan canonical form of the matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 4 \end{pmatrix}$$

**Problem 8.**
How many zeros does the function $f(z) = 3z^{100} - e^z$ have inside the unit circle (counting multiplicities)? Are the zeros distinct?

**Problem 9.**
Prove that $\cos 72°$ is algebraic and find its minimal polynomial.

**Problem 10.**
Let the function $f$ be analytic in the entire complex plane, and suppose that $f(z)/z \to 0$ as $|z| \to \infty$. Prove that $f$ is constant.

**Problem 11.**
Prove that any group of order 77 is cyclic.

**Problem 12.**
Suppose $f$ is a differentiable real valued function such that $f'(x) > f(x)$ for all $x \in \mathbb{R}$ and $f(0) = 0$. Prove that $f(x) > 0$ for all positive $x$.