**Pure Mathematics Qualifying Exam**
**September 2, 2006**

**Part I**

PROBLEM 1. Let $B$ be the $13 \times 13$ matrix whose entry in the $i$th row and $j$th column equals $i + j$. Let $V$ be the set of vectors $\mathbf{v} \in \mathbb{R}^{13}$ such that $B\mathbf{v} = \mathbf{0}$. Prove that $V$ is a subspace of $\mathbb{R}^{13}$, and calculate the dimension of $V$.

PROBLEM 2. (a) Show that every group of order 99 is the direct product of a group of order 9 and a group of order 11.

(b) Show that any group of order 9 is abelian. Conclude that every group of order 99 must be abelian.

(c) How many different groups of order 99 are there, up to isomorphism?

PROBLEM 3. Let $f(x)$ be a real-valued function defined on $[0,1]$ that is differentiable up to and including its endpoints. Give a proof of the following limiting value:

$$\lim_{n\to\infty} \left[ (n+1) \int_0^1 x^n f(x)\,dx \right] = f(1)\,.$$

PROBLEM 4. Find the image of the unit disk $\{z\colon |z| < 1\}$ under the mapping

$$w = f(z) = i\,\mathrm{Log}\left(\frac{i+z}{i-z}\right),$$

where Log denotes the principal value of the logarithm function. What effect would choosing a different branch of the logarithm function, rather than Log, have on your answer?

PROBLEM 5. Let $\mathbf{u} \in \mathbb{C}^n$ and $\mathbf{v} \in \mathbb{C}^n$ be column vectors, and consider the matrix $A$ defined by $A = I + \mathbf{u}\mathbf{v}^*$, where $I$ is the $n \times n$ identity matrix. Here $^*$ denotes conjugate transpose.

(a) Characterize the pairs of vectors $\mathbf{u}$ and $\mathbf{v}$ for which $A$ is singular.

(b) When $A$ is non-singular, show that its inverse is of the form $A^{-1} = I + \alpha\mathbf{u}\mathbf{v}^*$ for some scalar $\alpha$ (depending on $\mathbf{u}$ and $\mathbf{v}$). Determine an explicit expression for $\alpha$.

(c) When $A$ is singular, what is the nullspace of $A$?

(continued on back)

PROBLEM 6. In this problem, "irreducible polynomial" means "a polynomial with integer coefficients that is irreducible over $\mathbb{Q}$".

(a) Suppose $p(x)$ is an irreducible polynomial of degree $n$ and has two roots $r_1, r_2$ satisfying $r_1 r_2 = 5$. Prove that $n$ is even.

(b) If $p(x) = x^4 + ax^3 + bx^2 + cx + d$ is an irreducible polynomial of degree 4 with the property in part (a), prove that $d = 25$ and $c = 5a$.

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**Part II**

PROBLEM 1. For which $s \in \mathbb{R}$ does the infinite series

$$f(s) = \sum_{n=2}^{\infty} \frac{1}{n(\log n)^s}$$

converge? Give a careful proof of your result.

PROBLEM 2. The dilogarithm function is defined by $f(z) = \sum_{k=1}^{\infty} z^k/k^2$ on the unit disk. Show that $f$ can be extended to an analytic function on $\mathbb{C} \setminus [1, \infty)$. What is

$$\lim_{\varepsilon \to 0+} \big( f(2 + \varepsilon i) - f(2 - \varepsilon i) \big)?$$

Hint: show that

$$f(z) = -\int_0^z \log(1 - w) \frac{dw}{w}.$$

PROBLEM 3. How many different $4 \times 4$ matrices $M$, up to similarity, have the property that $M^4 = M^2$ but $M^3 \neq M$?

PROBLEM 4. Prove or disprove: (a) $\mathbb{Z}[x]$ is a principal ideal domain; (b) $\mathbb{Z}[x]$ is a unique factorization domain.

PROBLEM 5. Let $x_1, \ldots, x_N$ be real variables. Find the maximum value of the second symmetric function

$$s_2(x_1, \ldots, x_N) = \sum\sum_{1 \leq i < j \leq n} x_i x_j$$

subject to the constraints $x_1 \geq 0, \ldots, x_N \geq 0$ and $x_1 + x_2 + \cdots + x_N = 1$.

(continued on back)

PROBLEM 6. For any positive integer $N$, let $C_N$ denote the boundary (oriented in the counterclockwise direction) of the rectangle with vertices at

$$\left(N+\tfrac{1}{2}\right)(1+i),\ \left(N+\tfrac{1}{2}\right)(-1+i),\ \left(N+\tfrac{1}{2}\right)(-1-i),\ \text{and}\ \left(N+\tfrac{1}{2}\right)(1-i).$$

Define $I_N$ by

$$I_N = \int_{C_N} \frac{\pi}{z^2 \sin(\pi z)}\, dz\,.$$

(a) Prove directly that $I_N \to 0$ as $N \to +\infty$.

(b) By using the residue theorem and the result in part (a), prove the identity

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} = -\frac{\pi^2}{12}\,.$$