Applied Mathematics Qualifying Exam
September 2, 2006

Part I

PROBLEM 1. Let B be the 13×13 matrix whose entry in the ith row and jth column equals $i + j$. Let V be the set of vectors $\mathbf{v} \in \mathbb{R}^{13}$ such that $B\mathbf{v} = \mathbf{0}$. Prove that V is a subspace of $\mathbb{R}^{13}$, and calculate the dimension of V.

PROBLEM 2. Consider the Fourier sine series expansion of the function $f(x)$ defined by

$$f(x) = 1\,, \quad 0 \le x \le \pi\,.$$

Recall that the Fourier sine series has the form $f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$ for all $x \in \mathbb{R}$.

(a) Calculate the coefficients b_n in this Fourier sine series, and find an infinite series expansion for $\pi/4$.

(b) Let $S_{2N-1}(x) = \sum_{n=1}^{2N-1} b_n \sin(nx)$ denote the sum of the first $2N - 1$ terms in this Fourier sine series. Show that for all positive integers N,

$$S_{2N-1}(x) = \frac{2}{\pi} \int_0^x \frac{\sin(2Nu)}{\sin u}\, du\,.$$

(c) For a given positive integer N, use the result of part (b) to determine the smallest positive real number $x = x_N$ at which $S_{2N-1}(x)$ has a local maximum. How does this relate to the Gibbs phenomenon?

PROBLEM 3. Let $f(x)$ be a real-valued function defined on $[0, 1]$ that is differentiable up to and including its endpoints. Give a proof of the following limiting value:

$$\lim_{n\to\infty} \left[(n+1) \int_0^1 x^n f(x)\, dx \right] = f(1)\,.$$

PROBLEM 4. Find the image of the unit disk $\{z\colon |z| < 1\}$ under the mapping

$$w = f(z) = i\,\mathrm{Log}\left(\frac{i+z}{i-z}\right),$$

where Log denotes the principal value of the logarithm function. What effect would choosing a different branch of the logarithm function, rather than Log, have on your answer?

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PROBLEM 5. Let $\mathbf{u} \in \mathbb{C}^n$ and $\mathbf{v} \in \mathbb{C}^n$ be column vectors, and consider the matrix A defined by $A = I + \mathbf{u}\mathbf{v}^*$, where I is the $n \times n$ identity matrix. Here * denotes conjugate transpose.

(a) Characterize the pairs of vectors $\mathbf{u}$ and $\mathbf{v}$ for which A is singular.

(b) When A is non-singular, show that its inverse is of the form $A^{-1} = I + \alpha\mathbf{u}\mathbf{v}^*$ for some scalar α (depending on $\mathbf{u}$ and $\mathbf{v}$). Determine an explicit expression for α.

(c) When A is singular, what is the nullspace of A?

PROBLEM 6. Consider the following convection-diffusion equation for $u(x,t)$:

$$u_t + cu_x = Du_{xx}\,, \qquad 0 < x < \infty\,, \quad t > 0\,,$$
$$u(0,t) = f(t)\,, \qquad u(x,0) = 0\,, \qquad u \ \text{ bounded as } x \to +\infty\,.$$

Here $c > 0$ and $D > 0$ are constants.

(a) When $D = 0$ (no diffusion), find $u(x,t)$ using the method of characteristics.

(b) When $D > 0$, calculate the solution using Laplace transforms. Two relevant Laplace transform pairs are:

$$\mathcal{L}\left(e^{rt} f(t)\right) = F(s-r)\,, \quad F(s) = \mathcal{L}(f(t))\,;$$
$$\mathcal{L}^{-1}\left(e^{-\lambda\sqrt{s}}\right) = \frac{\lambda}{2\sqrt{\pi}t^{3/2}} e^{-\lambda^2/(4t)}\,, \quad \lambda > 0\,.$$

(c) Briefly discuss the main qualitative differences between the solutions for the case $D = 0$ and for the case $D > 0$ with regards to the speed of propagation of signals and the propagation of any discontinuities.

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Part II

PROBLEM 1. For which $s \in \mathbb{R}$ does the infinite series

$$f(s) = \sum_{n=2}^{\infty} \frac{1}{n(\log n)^s}$$

converge? Give a careful proof of your result.

PROBLEM 2. Consider the following initial value problem for $y(t)$ on $t \geq 0$:

$$y''' + 2y'' + 4y' + 6y = \sin(t),$$

with $y(0) = y'(0) = y''(0) = 0$. By taking the Laplace transform and then examining the roots of some polynomial $p(s)$ in the right half-plane $\mathrm{Re}(s) \geq 0$, prove that $y(t)$ is bounded as $t \to \infty$. In addition, find constants a and b such that $y(t) \sim a \sin t + b \cos t$ as $t \to \infty$.

PROBLEM 3. Given a positive integer K, consider the $2K \times 2K$ tridiagonal matrix $\mathcal{M}$ defined by

$$\mathcal{M} = \begin{bmatrix} -b & a & & & & & \\ a & -b-d & d & & & & \\ & d & -b-d & a & & & \\ & & a & -b-d & \ddots & & \\ & & & \ddots & \ddots & d & \\ & & & & d & -b-d & a \\ & & & & & a & -b \end{bmatrix}.$$

Here the real numbers a, b, and d satisfy $b > a > 0$ and $d > 0$. Prove that $\mathcal{M}$ is negative definite. (Hint: It may be convenient to decompose $\mathcal{M}$ into the sum of two block-diagonal matrices).

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PROBLEM 4. Consider the following nonlinear system of ODE's for $x = x(t)$ and $y = y(t)$:

$$x' = x - y - x^3, \qquad y' = x + y - y^3.$$

By first converting this system to polar coordinates, prove that there exists a periodic solution of this system inside the annulus $1 < r < \sqrt{2}$, where $r = \sqrt{x^2 + y^2}$.

PROBLEM 5. Let $x_1, \ldots, x_N$ be real variables. Find the maximum value of the second symmetric function

$$s_2(x_1, \ldots, x_N) = \sum\sum_{1 \le i < j \le n} x_i x_j$$

subject to the constraints $x_1 \ge 0, \ldots, x_N \ge 0$ and $x_1 + x_2 + \cdots + x_N = 1$.

PROBLEM 6. For any positive integer N, let C_N denote the boundary (oriented in the counterclockwise direction) of the rectangle with vertices at

$$\left(N + \tfrac{1}{2}\right)(1 + i), \ \left(N + \tfrac{1}{2}\right)(-1 + i), \ \left(N + \tfrac{1}{2}\right)(-1 - i), \text{ and } \left(N + \tfrac{1}{2}\right)(1 - i).$$

Define I_N by

$$I_N = \int_{C_N} \frac{\pi}{z^2 \sin(\pi z)} \, dz.$$

(a) Prove directly that $I_N \to 0$ as $N \to +\infty$.

(b) By using the residue theorem and the result in part (a), prove the identity

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} = -\frac{\pi^2}{12}.$$